MOTION OF THE SIMPLE PENDULUM

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I. INTRODUCTION

Is the simple pendulum isochronal? Asked another way, does its period depend on anything other than its length? I want to answer this question by setting out a mathematically precise description of the motion of a simple pendulum, use this description to evaluate its period, compare this exact description to other, probably more familiar, descriptions that are approximately correct for small swing angles, and then examine other factors that can influence the motion of the simple pendulum. The material presented here is not new. It can be found in many intermediate level physics textbooks. An overview of the physics of the pendulum can be found in the book by Rawlings (Ref. 1). A more detailed assessment of various factors that influence the motion of the pendulum is given in a recent article by Nelson and Olsson (Ref. 2).

A simple pendulum consists of a dimensionless (point) bob of mass M at the end of a massless, rigid rod of length l that swings freely from a suspension point that does not move. Gravity is the only external force acting on the pendulum. We assume that the gravitational force doesn't vary in the course of the swing of the pendulum, and that the Coriolis force (that tends to move the pendulum at right angles to its motion) is not an important factor governing the motion of the pendulum. All frictional forces are assumed to be small enough not to affect appreciably the motion caused by the gravitational force. Likewise, elastic forces involving a suspension spring are not included; nor are many other very real forces. In Section V, mention is made of the impact of such factors as temperature, humidity, etc.

The force on the mass M is the gravitational force, Mg. The acceleration g due to gravity is usually taken to be 32.2 ft/sec/sec, which equals 386 in/sec/sec, and, as shown in Sec.V, g does vary slightly with location on the Earth.

The pendulum shown in Fig. 1 is given an initial push that causes it to swing from the equilibrium position, a = 0, to the right to the maximum angle A, called the amplitude of the swing. The gravitational force always tends to pull the pendulum back to the vertical equilibrium position, a = 0, but the motion of the pendulum takes it to the left to -A before reversing direction. Physicists refer to the time

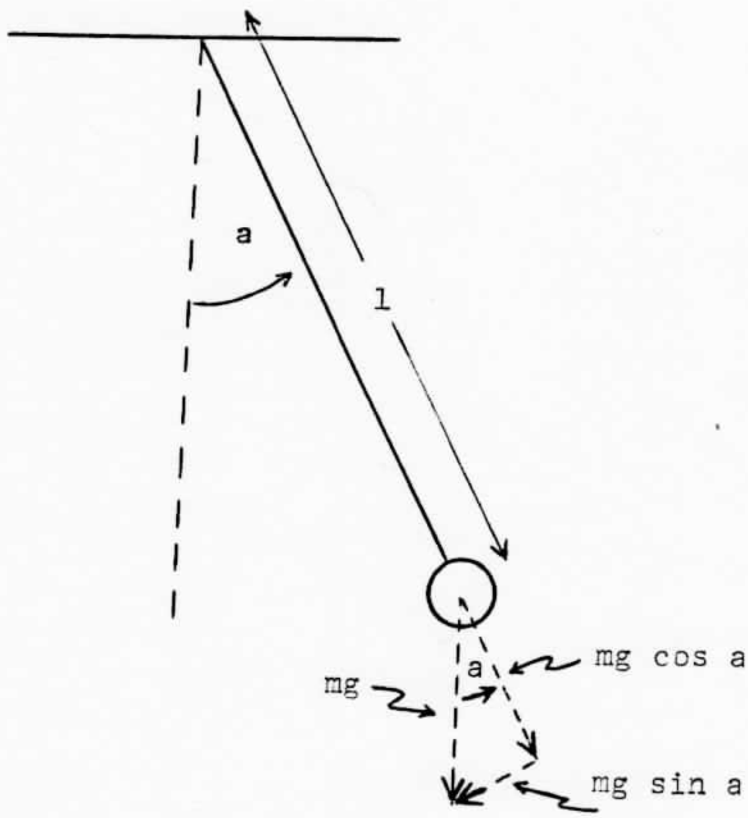


Figure 1

The simple pendulum with associated force vector diagram. The only force present is gravity, i.e. no friction. The pendulum rod doesn't flex, expand, or contract. It behaves like a perfectly straight string, i.e. no suspension spring. The mass of the bob is concentrated at a point at the end of the rod. Contrary to common belief, the resulting motion is not sinusoidal.

needed for a complete swing from a = 0 to +A to 0 to -A to 0 to be the period of the pendulum. Horologists refer to this as twice the beat of the pendulum. So, a pendulum that has a period of two seconds is said to beat seconds.

The dependence of a on time t and the dependence of the period T on l that are the most familiar:

$$a = A \sin(2\pi t/T) \tag{1}$$

$$T = 2\pi(l/g)^{1/2} \tag{2}$$

where $\pi \cong 3.141593$, are, as is shown in Sec. II, only approximate solutions to the exact equations of motion.

The next section describes the development of the exact solutions to the equations of motion. Section III shows the origin of the approximate solutions, Eqs. (1) and (2). Section IV shows how accurately the approximate solutions represent the exact solutions. Section V shows the relative importance of other factors that affect the motion of the simple pendulum.

II. EXACT MOTION OF THE SIMPLE PENDULUM

The gravitational force, Mg, is a vector, always pointing toward the center of the earth. This vector quantity can be represented by the vector sum of two components: one, Mg cos a, pointing in the direction of the pendulum, and one, Mg sin a, pointing perpendicular to the pendulum. These two components are indicated by the dotted arrows in Fig. 1. It is the component Mg sin a, perpendicular to the rod, that is responsible for the motion of the pendulum.

A torque N exerted on a rotating object having a moment of inertia I results in the radial acceleration d^2a/dt^2:

$$N = I\, d^2a/dt^2 \tag{3}$$

The torque on the pendulum is the product of: the component of the gravitational force that is perpendicular to the pendulum rod, -Mg sin a, and the radius arm l through which the force acts to produce the torque. The negative sign means that the force is always directed towards the equilibrium point, a = 0.

The moment of inertia of the pendulum assembly is $I = Ml^2$. Inserting this and N = -Mgl sin a in Eq. (3) results in:

$$Ml^2 d^2a/dt^2 = - Mgl \sin a \quad (4)$$

Dividing both sides of this equation by Ml^2 gives what I call the First Equation of Motion of the Simple Pendulum:

$$d^2a/dt^2 = - (g/l) \sin a \quad (5)$$

The solution to this equation gives a as a function of time t, from which the period of swing can be found. Notice that the mass of the pendulum does not appear in Eq. (5). Because of this, a and the period do not depend on the mass of the pendulum. Consistent with our assumptions, the motion of the pendulum is the same whether the mass of the bob is large or small. The mass does not affect the motion. The reason for this is that the rod is massless, and the mass of the bob is concentrated in a point. If the rod has appreciable mass compared to the bob, then the mass of each would affect the period. This is elaborated on in Sec. V. The assumptions we made are that the only force acting on the pendulum is gravity, so frictional forces and elastic forces are unimportant. Another assumption is that the value of g does not change at the location of the bob as the bob goes through its motion.

A second equation governing the motion of the pendulum is obtained from the physics principle known as the Conservation of Energy, which states that the total energy of an object--the sum of its kinetic energy and its potential energy--does not vary. The kinetic energy is $(1/2)Mv^2$, where v is the velocity. In these coordinates, the kinetic energy of the pendulum is $(1/2)M(l\,da/dt)^2$. The potential energy is Mgh, where h is the height above some arbitrary reference height. For convenience, let this reference height be at the suspension point of the pendulum. Then the potential energy is $- Mgl \cos a$, and the total energy of the pendulum is

$$(1/2)\ Ml^2(da/dt)^2 - Mgl \cos a \quad (6)$$

Conservation of Energy requires that the total energy be constant. So the total energy of the pendulum is the same when the pendulum is at any angle a and when the pendulum is at the angle a (=A, the maximum angle of swing, called the amplitude) for which da/dt = 0. Equating the total energy at these two angles gives

$$(1/2)\ Ml^2(da/dt)^2 - Mgl \cos a = -Mgl \cos A \quad (7)$$

Dividing both sides of this equation by $(1/2)Ml^2$, and rearranging terms, gives what I call the Second Equation of Motion of the Simple Pendulum:

$$da/dt = (2g/l)^{1/2} [\cos a - \cos A]^{1/2} \tag{8}$$

Differentiating this with respect to time gives Eq. (5), the First Equation of Motion of the Simple Pendulum. Like the First Equation, the Second Equation is independent of the mass of the pendulum.

The solution to the First and Second Equations of Motion, Eqs. (5) and (8), involving the dependence of a on t, cannot be expressed in terms of simple functions. Nevertheless, the dependence of a on t can be obtained in terms of tabulated functions. The solution to Eq. (8) is found by making these substitutions: $\cos a = 1 - 2\sin^2 a/2$, and

$$\sin q = \sin (a/2) / \sin (A/2) \tag{9}$$

After much simplification, the integration of Eq. (8) results in

$$(g/l)^{1/2} t = \int_0^{\arcsin[\sin(a/2)/\sin(A/2)]} [1 - \sin^2(A/2) \sin^2 q]^{-1/2} dq \tag{10}$$

This is in the form of an Elliptic Integral of the First Kind (see Ref. 3), which is defined by:

$$F(u\backslash v) = \int_0^u (1 - \sin^2 v \sin^2 x)^{-1/2} dx \tag{11}$$

Using this definition, the exact solution to the equations of motion of the pendulum is

$$(g/l)^{1/2} t = F(\arcsin \left[\frac{\sin (a/2)}{\sin (A/2)}\right] \backslash A/2) \tag{12}$$

Although F(u\v) depends on u and v in a complex fashion, it is no different than any other function that depends on one or more variables. F(u\v) depends on the two variables, u and v, much the same as sin a depends on the one variable a. Both F(u\v) and sin a are quantities that can't be calculated easily, but must be looked up in tables of values. These tables of values of F(u\v) are reproduced in Appendix A from Ref. 3.

Figure 2 shows plots of a vs. $(g/l)^{1/2}t$ for various values of A. As we could anticipate, a increases from 0 at t = 0 to a maximum of A at 1/4 the period, and returns through

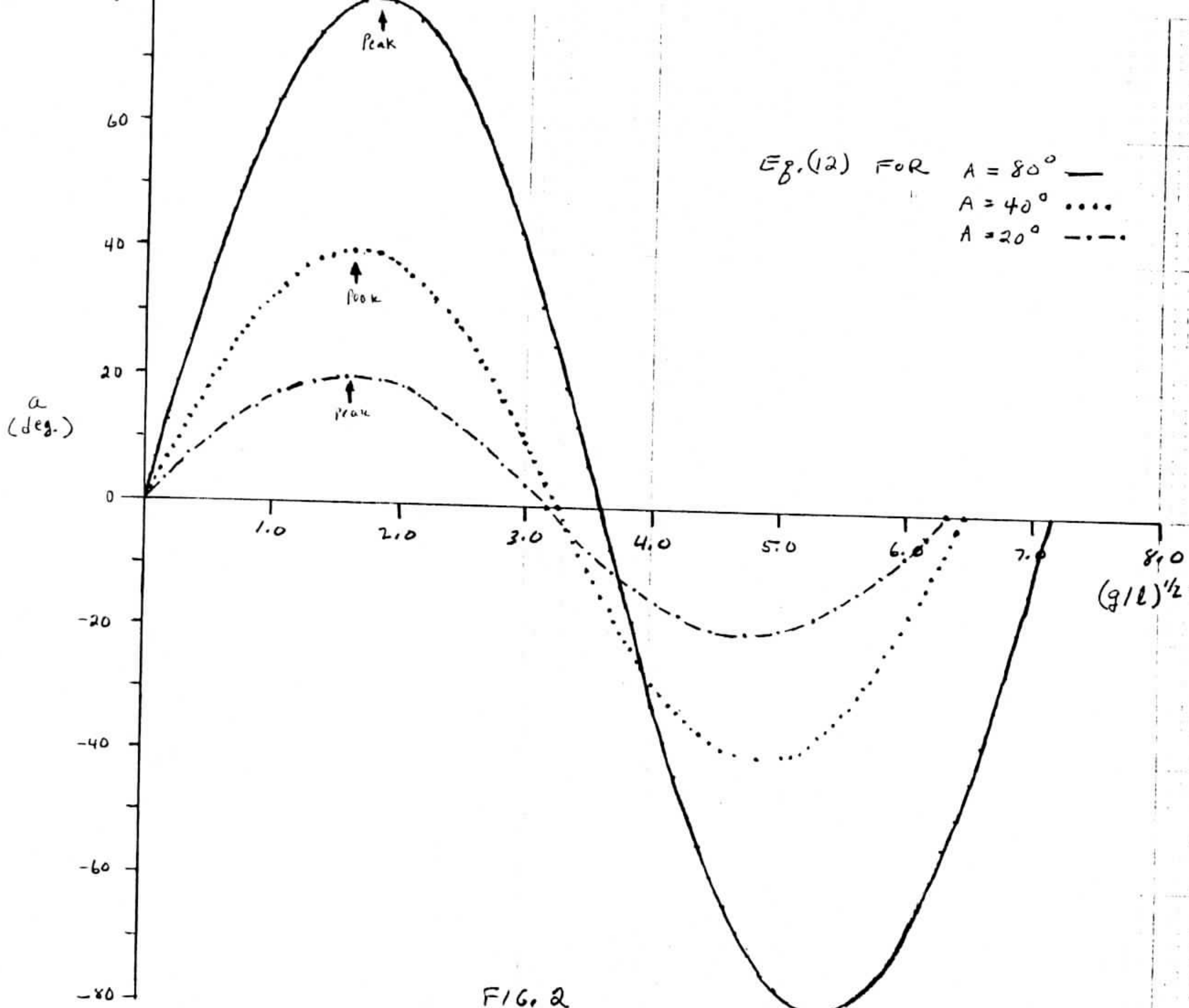

Peak
Peak
Peak
60
40
20
0
-20
-40
-60
-80
a (deg.)
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
$(g/l)^{1/2}t$
Eq.(12) FOR A = 80°
A = 40°
A = 20°

FIG. 2

a = 0 at 1/2 the period, to -A at 3/4 the period, and to a = 0 at the end of a full period. The shapes of the curves are reminiscent of sine curves, but they are not precisely the same, as is shown in Sec. IV.

If we carry out the integration in Eq. (10) from 0 to 90 degrees, the corresponding time runs from zero to 1/4 the period. So the exact solution for the period of the pendulum is

$$(1/4)(g/l)^{1/2}\, T = \int_0^{\pi/2} [1 - \sin^2(A/2)\, \sin^2 q]^{-1/2}\, dq \qquad (13)$$

Here we have introduced radian measure instead of degrees for the angle that represents the upper limit of the integral. In radian measure, π radians corresponds to 180 degrees. The conversion from degrees to radians is by

$$\text{angle in degrees} / 180 = \text{angle in radians} / \pi \qquad (14)$$

For example, 23 degrees corresponds to 0.401 radians.

Equation (13) is a simpler form of the Elliptic Integral of the First Kind because it depends on one less variable. It is in the form known as the Complete Elliptic Integral of the First Kind, K(z) (tables of values of K(z) are given in Appendix B, taken from Ref. 3):

$$K(m) = \int_0^{\pi/2} [1 - m \sin^2 q]^{-1/2}\, dq \qquad (15)$$

K(m) is the same as $F(\pi/2 \setminus \arcsin m^{1/2})$. If we multiply Eq. (13) by $4(l/g)^{1/2}$, we get an equation for the period of the pendulum:

$$T = 4(l/g)^{1/2}\, K[\sin^2(A/2)] \qquad (16)$$

Notice that the period of the simple pendulum depends on three, and only three, factors: l, g, and A.

The period of the pendulum increases as l increases. It is not in the same measure, though. Because T is proportional to the square root of l, T increases slower than l. This dependence of T on l is shown in Fig. 3 for a very small amplitude of swing, and for the largest possible amplitude of swing, A = 90 degrees. The figure lists

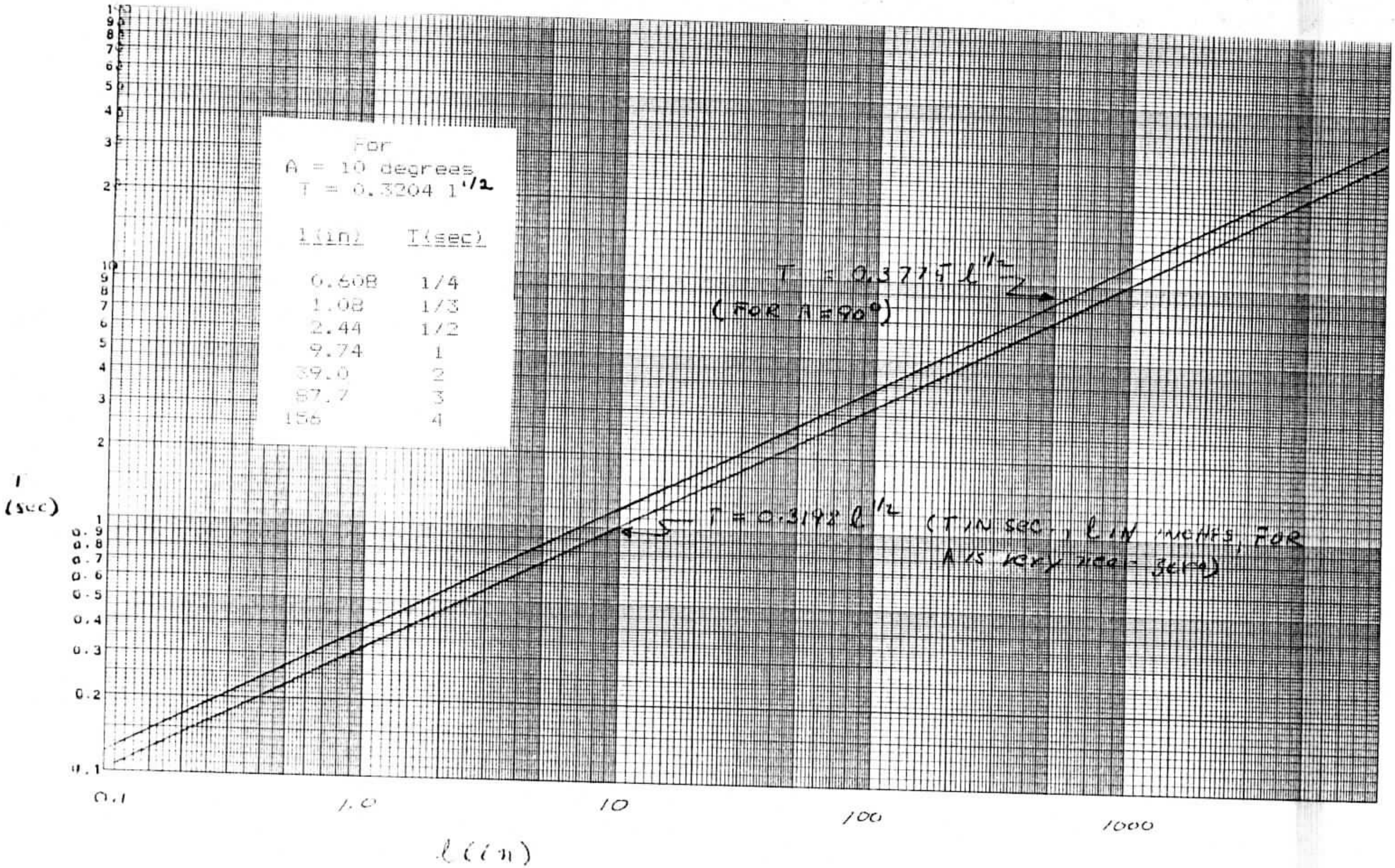

For
A = 10 degrees
T = 0.3204 l^1/2
l(in) T(sec)
0.608 1/4
1.08 1/3
2.44 1/2
9.74 1
39.0 2
87.7 3
156 4
T = 0.3775 l^1/2
(FOR A = 90°)
T = 0.3192 l^1/2 (T IN sec., l IN inches, FOR A IS very near zero)
T (sec)
l (in)
0.1
1.0
10
100
1000

FIG. 3

selected pairs of corresponding values of T and l for a pendulum swing of A = 10 degrees. Because T is proportional to the square root of l, doubling the length of the pendulum increases the period of the pendulum by a factor of the square root of 2, about 1.414. Notice that a 2-second pendulum (one that beats seconds) is about one meter in length (1 meter = 39.37 inches). A pendulum that beats with your pulse (72 beats per minute) has a length of about twenty seven inches.

The period of the pendulum also depends on g. The value of g changes with latitude and with altitude above sea level, although these changes are small. Section V shows how these small changes in g influence the motion of the pendulum.

The dependence of T on A is contained in the term $K[\sin^2(A/2)]$ in Eq. (16). If we divide both sides of Eq. (16) by $2\pi(l/g)^{1/2}$, we get

$$T/\ 2\pi(l/g)^{1/2} = (2/\pi)\ K[\sin^2(A/2)] \tag{17}$$

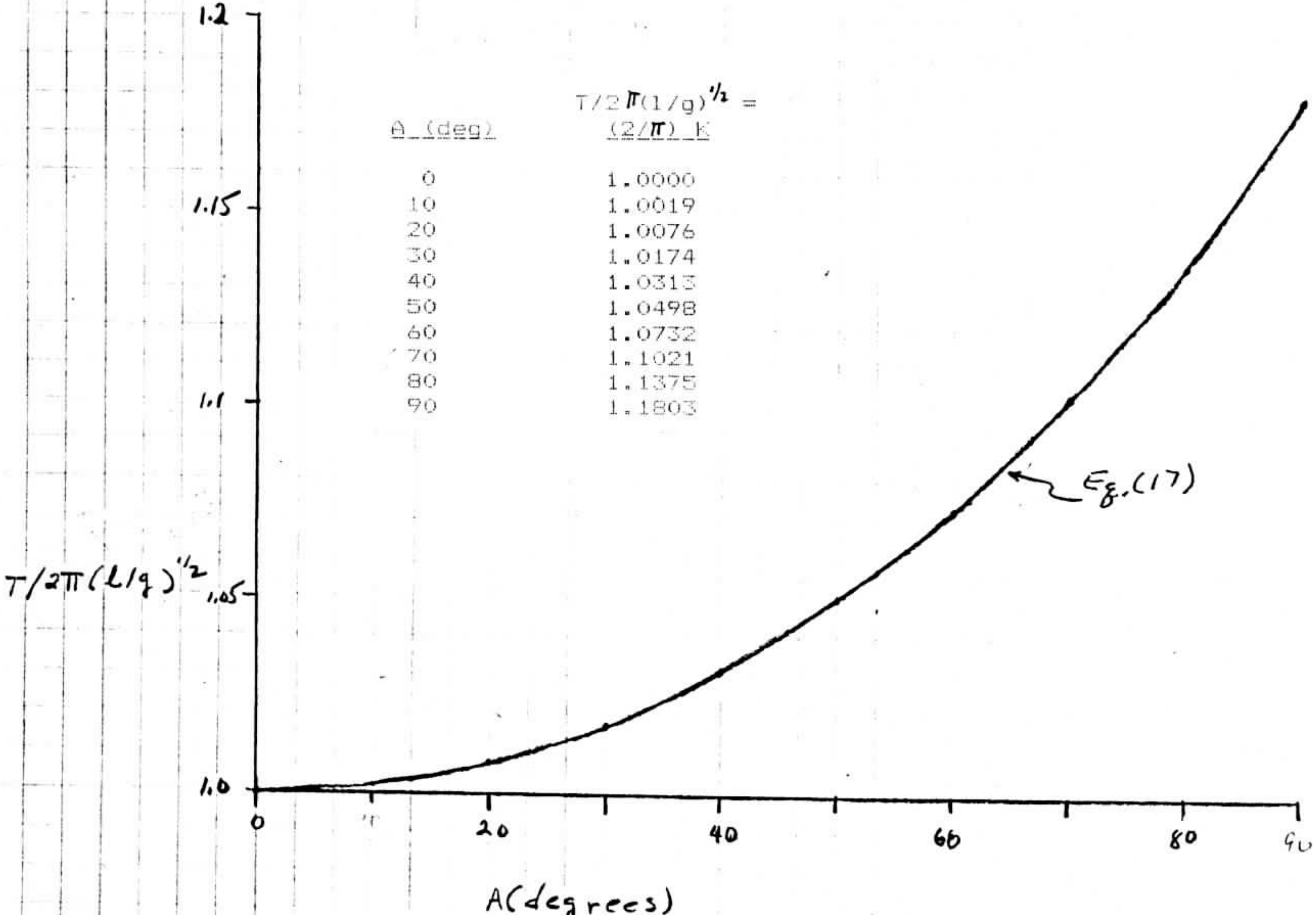
The quantity $2\pi(l/g)^{1/2}$ is Eq. (2), the equation for the period when the amplitude of swing, A, is small. Thus, the right side of Eq. (17) shows the error in T, as calculated by Eq. (2). A plot of Eq. (17) is shown in Fig. 4. When A is small, T approximately equals $2\pi(l/g)^{1/2}$. The exact period is within 1% of $2\pi(l/g)^{1/2}$ for A < 22.79 degrees. For purposes of comparison, I measured the values of A for several clocks. An Ogee had A = 6 degrees, an English bracket clock, circa 1880, had A = 1.3 degrees. The pendulums of each of these clocks, of course, don't meet the requirements we set for the definition of a simple pendulum: these pendulums don't swing freely; they are impulsed on each beat by a crutch or a crown; they operate under non-ideal conditions, and they experience frictional forces.

Values of the Complete Elliptic Integral of the First Kind can be found from the table in Appendix B. Alternatively, they can be calculated approximately from the following equation (Ref. 3):

$$K(m) = (\pi/2)\ [1 + (1/2)^2 m + (1\cdot 3/2\cdot 4)^2 m^2 + (1\cdot 3\cdot 5/2\cdot 4\cdot 6)^2 m^3 + \ \dots\ ] \tag{18}$$

The series of three dots indicate there are an infinity of other terms that follow these four, all in a form that can be inferred from the previous terms.

Using this expansion series for K(m), Eq. (16) for the period of the simple pendulum becomes:



A (deg)	$T/2\pi(1/g)^{1/2} = (2/\pi)\ K$
0	1.0000
10	1.0019
20	1.0076
30	1.0174
40	1.0313
50	1.0498
60	1.0732
70	1.1021
80	1.1375
90	1.1803

FIG 4

$$T = 2\pi(l/g)^{1/2}\,[1 + (1/4)\sin^2(A/2) + (9/64)\sin^4(A/2) + (25/256)\sin^6(A/2) + \ldots] \qquad (19)$$

Good accuracy can be obtained from using just the first few terms. For accuracy to within 1%, only the first term is needed up to 22.79 degrees of swing.

III APPROXIMATE MOTION OF THE SIMPLE PENDULUM

Equation (16) is the exact equation for the period of the simple pendulum. For small swing amplitudes, Eq. (16) is well approximated by Eq. (2). Equation (2) is probably more familiar than is Eq. (16). The approximate equation can be derived by the following analysis, which begins with The First Equation of Motion of the pendulum

$$d^2a/dt^2 = -\,(g/l)\sin a \qquad (20)$$

The presence of sin a is the source of the exact, but complex expression, Eq. (16), for the period of the pendulum. If the swing of a pendulum is small, on the order of a few degrees, an approximation to sin a can be used that greatly simplifies the derivation and the result.

If a is in radian measure, the Fourier Series expansion of sin a is

$$\sin a = \sum_{n=0}^{\infty} (-1)^n a^{2n+1}/(2n+1)! = a - a^3/3! + a^5/5! - a^7/7! + \ldots \qquad (21)$$

where $n! = n(n-1)(n-2)\ldots 1$. If a is very much smaller than one, $a^3/3!$ is much smaller than a, and the higher-order terms are even smaller still. For such situations, the small-angle approximation

$$\sin a \simeq a \qquad (22)$$

is accurate (see the table below).

a (deg)	sin a	a (rad)	$a - a^3/3!$	$a - a^3/3! + a^5/5!$
0	0	0	0	0
1	.01745	.01745		
3	.05234	.05236	.05234	
5	.08716	.08727	.08716	
7	.12187	.12217	.12187	
10	.17365	.17453	.17364	
20	.34202	.34907	.34198	
30	.50000	.52360	.49968	

40	.64279	.69813	.64142	.64280
50	.76604	.87266	.76190	.76612

This small-angle approximation is accurate to 1% for angles up to a = 13.98 degrees. For the values of A for the clocks mentioned above, this approximation is sufficiently accurate (as will be shown in the next section) that we can rewrite the First Equation of Motion of the pendulum, Eq. (20), as

$$d^2a/dt^2 = -(g/l)\ a \qquad (23)$$

A solution to this is

$$a = A \sin[(g/l)^{1/2} t] \qquad (24)$$

as can be found by taking the second time derivative of a. As $(g/l)^{1/2}t$ increases from zero to 2π, a completes a full period, so that

$$(g/l)^{1/2}\ T = 2\pi \qquad (25)$$

Dividing both sides of this equation by $(g/l)^{1/2}$, we find that the period is

$$T = 2\pi(l/g)^{1/2} \qquad (26)$$

This is the expression for the period of a simple pendulum, Eq. (2), with which we are most familiar. It is only accurate for small-angle amplitude swings, though.

For small-amplitude swings, the Second Equation of Motion of the pendulum, Eq. (8), also gives Eqs. (24) and (26). To see that this is so, the first two terms of the Fourier Series expansion (0! = 1)

$$\cos a = \sum_{n=0}^{\infty} (-1)^n\ a^{2n}/(2n)! \qquad (27)$$

namely,

$$\cos a \simeq 1 - a^2/2 \qquad (28)$$

are substituted in Eq. (8) for cos a and cos A. The result is

$$da/dt = (g/l)^{1/2} (A^2 - a^2)^{1/2} \qquad (29)$$

Making the substitution r = a/A , and rearranging terms gives

$$(g/l)^{1/2} t = \int_0^{a/A} (1 - r^2)^{1/2} \, dr \tag{30}$$

This can be integrated with the substitution r = sin y to give

$$(g/l)^{1/2} t = \arcsin(a/A) \tag{31}$$

which can be rearranged into the more familiar form

$$a = A \sin[(g/l)^{1/2} t] \tag{32}$$

This is the same as the solution found from the First Equation of Motion.

IV COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS

With a little mathematical rearranging, the exact equation for the period, Eq. (16), can be written as

$$T = 2\pi (E/g)^{1/2} \tag{33}$$

where I introduce the "effective length", E, of the pendulum. The "effective length" is

$$E = l\{(2/\pi) K[\sin^2(A/2)]\}^2 \tag{34}$$

This means that the period of swing of any pendulum can be written in the approximate form of Eq. (2), and still give the exact result, provided that l is replaced by E. That is, a wide-swing pendulum has a period that corresponds to that of a small-swing pendulum, but with a different, longer length. Figure 5 shows a plot of E/l as a function of A. In the same way that the period depends on A, the ratio E/l increases as A increases, with the ratio increasing from 1.0 at A = 0 to 1.3932 at A = 90 degrees.

For example, a pendulum of length 9.74 inches has a period of 1 second for small-amplitude swings. If it is swung at an amplitude of 60 degrees, its new period is 1.0732 seconds (see Fig. 4). Using Eq. (2), this new period corresponds to that of a small-amplitude pendulum of 11.28 inch length. So we can say that for a 9.79 inch pendulum with a swing amplitude of 60 degrees, the pendulum's effective length is 11.28 inches.

Figure 6 shows the relationship between a and t for A = 40 degrees and A = 80 degrees. On the figure are two pairs

A (deg)	E/l From Eq. (34)
0	1.0000
10	1.0038
20	1.0153
30	1.0351
40	1.0636
50	1.1021
60	1.1518
70	1.2147
80	1.2939
90	1.3932

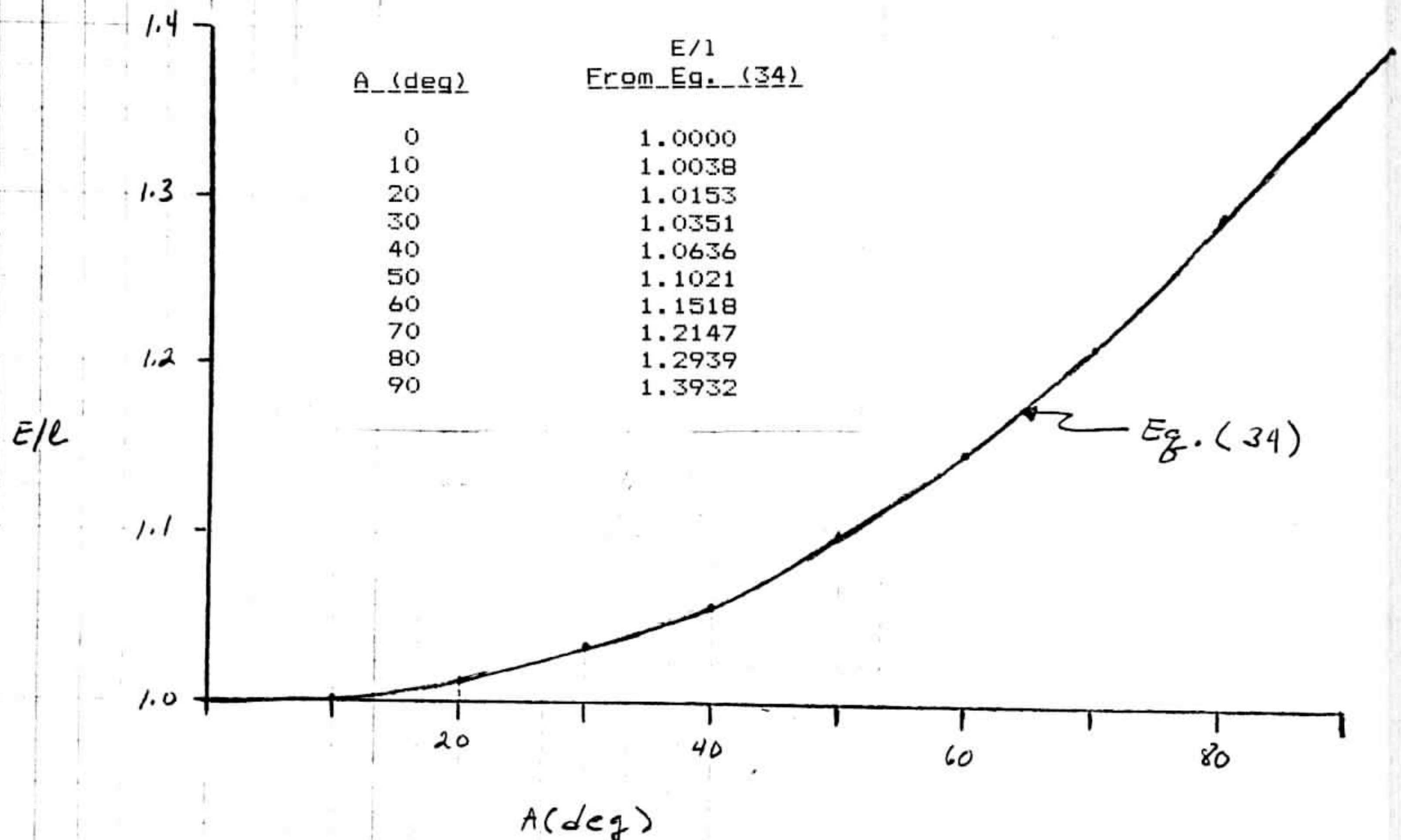
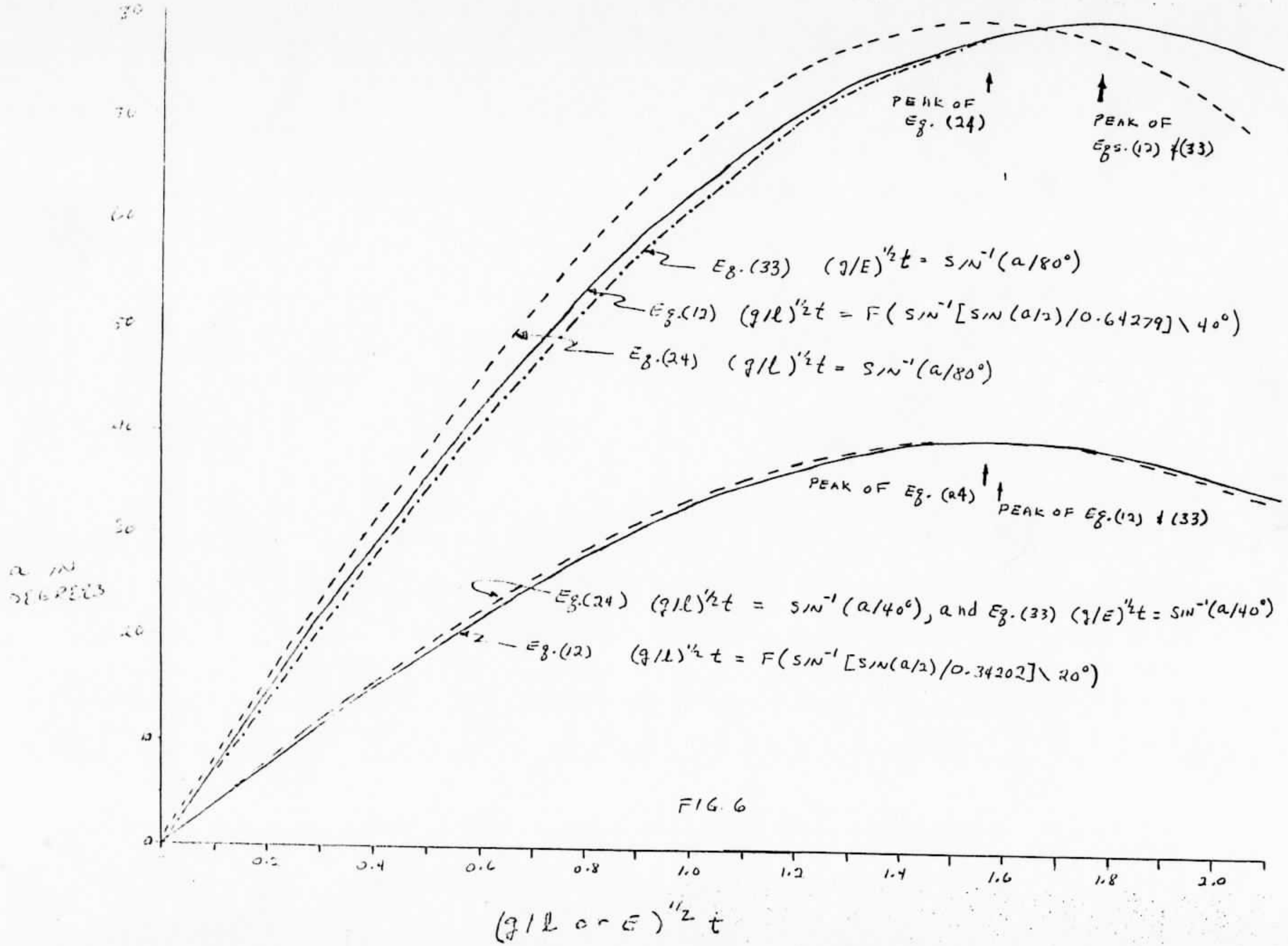


FIG. 5



PEAK OF Eg. (24)
PEAK OF Egs. (12) & (33)
Eg. (33) (g/E)^½ t = SIN⁻¹(a/80°)
Eg. (12) (g/l)^½ t = F(SIN⁻¹[SIN(a/2)/0.64279] \ 40°)
Eg. (24) (g/l)^½ t = SIN⁻¹(a/80°)
PEAK OF Eg. (24)
PEAK OF Eg. (12) & (33)
Eg. (24) (g/l)^½ t = SIN⁻¹(a/40°), and Eg. (33) (g/E)^½ t = SIN⁻¹(a/40°)
Eg. (12) (g/l)^½ t = F(SIN⁻¹[SIN(a/2)/0.34202] \ 20°)
a IN DEGREES
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
(g/l or E)^½ t

FIG. 6

of three curves (two of which are so close together on the figure as to be indistinguishab;e from each other). The solid lines show the exact relationship of a vs. t, as calculated from Eq. (12). The dashed lines show a vs. t for the small-angle result, as calculated from Eq. (24). The dash-dot lines show the small-angle result, but with l replaced by E, as given by Eq. (34).

As can be seen from these curves, use of the small-angle equation always results in a period that is too small. Because of this, the small-angle equation gets 'out of synch' with the exact result after several periods. This synchronization problem is remedied by using the small-angle equation with l replaced by E, but this equation consistently under-estimates the exact value of a. In general, though, it is a more accurate representation than is the small-angle equation without the use of E.

Frictional wear tends to lessen the amplitude of swing. This changes the period of the pendulum in accordance with Eq. (16). If a pendulum could be devised that would shorten itself as it swings outward and lengthen itself as it returns to a = 0 in the correct fashion, then the period of the pendulum would not depend on its amplitude. This is the concept behind the cycloidal 'cheeks' invented by Christiaan Huygens (see Refs. 1, 5, and 6).

Huygens devised a pair of cheeks fixed to the suspension point (see Fig. 7) that cause the upper end of the suspension spring to wrap around the cheeks, thus shortening the length of the pendulum and keeping the pendulum isochronous even if the amplitude of swing changes. Huygens discovered that the proper shape of these cheeks is cycloidal, with an axis of the cycloid being one half the length of the pendulum. This is a truly remarkable discovery. A direct consequence of this is that if you make a bowl whose cross section is cycloidal, and place a marble on the surface, the marble always takes the same time to get to the bottom, no matter where the marble begins, whether near the bottom, or six feet from the bottom!

The Complete Elliptic Integral of the First Kind, K(m), can be expressed as the infinite series given as Eq. (18), which allows the calculation of K, hence T, without the need to resort to Appendix B. For small angles, $m = \sin^2(A/2)$ is close to zero, and the terms in m^2, m^3, etc. are much smaller than the term in m, so

$$K(m) \simeq (\pi/2)\ [\ 1 + m/4] \qquad (35)$$

Correspondingly,

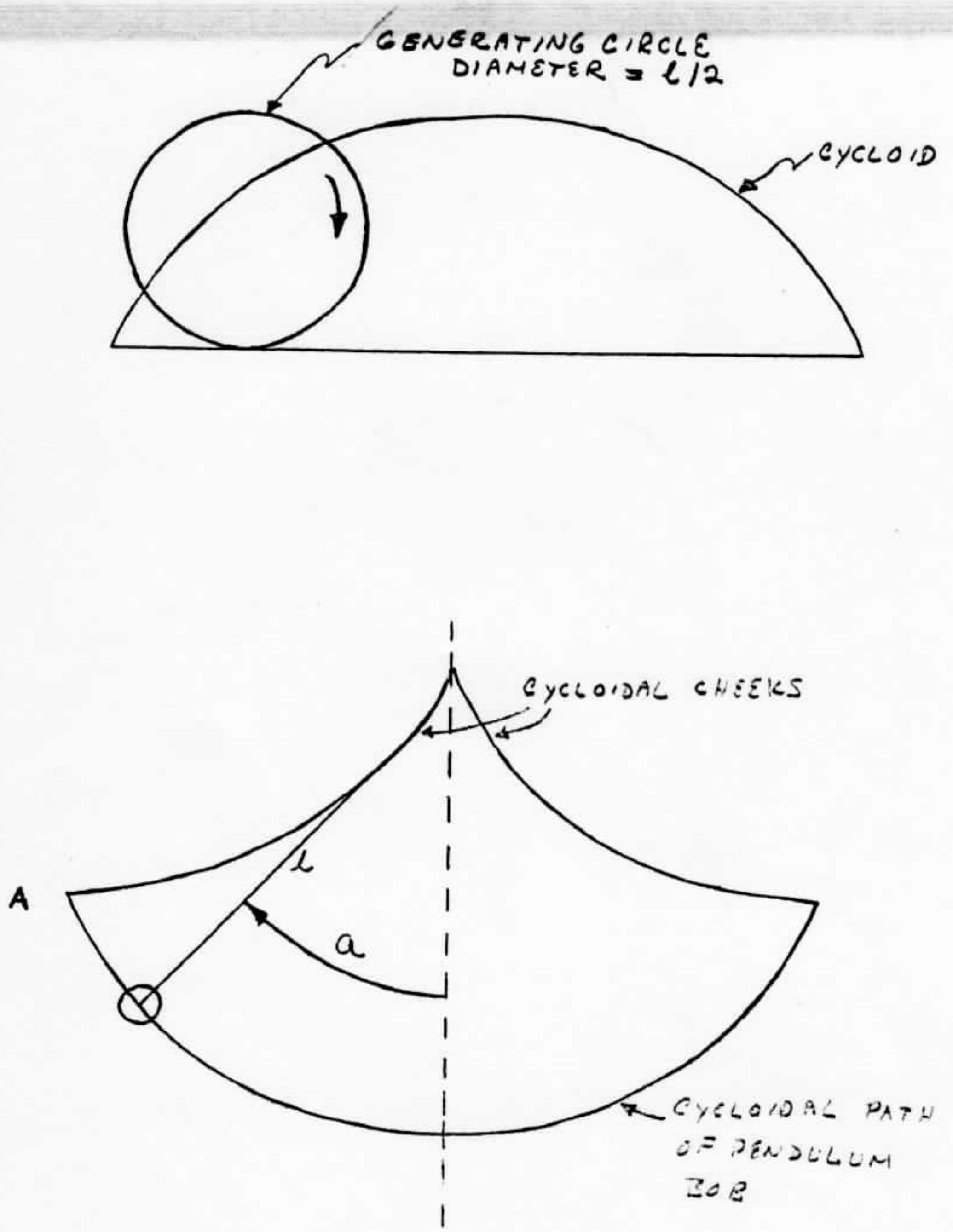

GENERATING CIRCLE
DIAMETER = l/2
CYCLOID
CYCLOIDAL CHEEKS
A
l
a
CYCLOIDAL PATH
OF PENDULUM
BOB

FIG. 7

$$T \simeq 4(l/g)^{1/2} (\pi/2) [1 + (1/4) \sin^2 (A/2)] \quad (36)$$

and using the small-angle approximation $\sin (A/2) \simeq A/2$

$$T \simeq 2\pi (l/g)^{1/2} [1 + A^2/16] \quad (37)$$

The accuracy of this equation is indicated in the table below.

A (degrees)	$T/2\pi(l/g)^{1/2}$ exact	$T/2\pi(l/g)^{1/2}$ Eq. (37)
0	1.0000	1.0000
10	1.0019	1.0019
20	1.0076	1.0076
30	1.0174	1.0171

V SOME FACTORS AFFECTING THE SIMPLE PENDULUM'S MOTION

As described in the previous sections, in the absence of any forces other than gravity, the only factors influencing the motion of a simple pendulum are: (1) its length l, (2) the gravitational acceleration g, and (3) the amplitude of swing A. These three factors are influenced by environmental conditions. Temperature and humidity change the length of materials from which pendulum rods are made. Latitude and elevation affect g. Barometric pressure, friction, humidity, and ease of air flow around a non-zero sized bob can affect A.

V a. The Effect of Changing a Pendulum's Length

Using the exact equation for the period, Eq. (16),

$$T = B(l/g)^{1/2} \quad (38)$$

where $B = 4 K[\sin^2(A/2)]$. The value of B depends only on the value of A. It does not depend on l or g. Differentiating T in Eq. (38) with respect to l gives

$$dT/dl = B(4lg)^{-1/2} \quad (39)$$

Dividing this by Eq. (38) and rearranging, gives

$$dT/T = (1/2)\ dl/l \quad (40)$$

This equation gives the relation between the fractional change in the period of a pendulum and the fractional change in the length of the pendulum. Notice that a given percentage change in l, dl/l, results in only half that percentage change in T. A 2% increase in the length of a pendulum gives a 1% increase in the period.

Equation (40) is useful for regulating pendulum clocks. If you observe that your pendulum clock is gaining one hour a day (dT/T = 1/24), you need to lengthen the pendulum by 8 1/3 % (dl/l = 2 dT/T = 1/12 = 8 1/3 %). A more practical method of regulating pendulum clocks is given in Ref. 4, which relates the time gained or lost by a pendulum clock to the number of turns needed to be taken on the rating nut to bring the clock into regulation.

Materials expand and contract as their temperatures vary. For this reason, pendulum rods made from a single material vary in length as the temperature varies. Although the linear thermal coefficient of expansion, w, is different for each material, most materials expand with increasing temperature in accordance with

$$l = L\ (1 + wC) \tag{41}$$

where L is the length of the material at zero degrees Centigrade, and C is the temperature. The value of w for any material depends on the grade and alloy of that material. Typical steels, for example, can have values as small as $9.07 \times 10^{-6}\ (^{\circ}C)^{-1}$, or as large as $1.21 \times 10^{-5}\ (^{\circ}C)^{-1}$. The table below presents values of w that represent averages of different grades and alloys. The values are appropriate for temperatures in the range of 10 degrees C to 90 degrees C. Outside this range, the values of w are different.

Material	$w\ (^{\circ}C)^{-1}$ **
Aluminum	2.3×10^{-5}
Brass	1.9×10^{-5}
Copper	1.7×10^{-5}
Glass	8.3×10^{-6}
Invar*	7×10^{-7}
Iron	1.1×10^{-5}
Mercury***	6×10^{-5}
Steel	1.1×10^{-5}
Wood	6×10^{-6}

* (Nickel Steel, 36 % Nickel)

** (Values taken from Ref. 7)

*** The cubical expansion of liquid mercury is V = V (1 + αC), with $\alpha = 1.8 \times 10^{-4}\ (^{\circ}C)^{-1}$ (Ref. 7). Correspondingly, the linear thermal coefficient of expansion, one third the cubical coefficient, is $w = 6 \times 10^{-5}\ (^{\circ}C)^{-1}$.

The linear coefficient of expansion for wood is different for wood cut with the grain than it is for wood cut against the grain. Pine cut with the grain has $w = 5.4 \times 10^{-6}$ $(^oC)^{-1}$, and pine cut against the grain has $w = 3.4 \times 10^{-5}$ $(^oC)^{-1}$. Cut parallel to the grain, the values of w for various woods range from $2.6 \times 10^{-6} (^oC)^{-1}$ for beech to $9.5 \times 10^{-6} (^oC)^{-1}$ for ash.

Notice that all these values of w are on the order of $10^{-5} (^oC)^{-1}$. This means that a temperature increase of 10 degrees Centigrade produces an increase in length of one tenth of a millimeter per meter of length. Invar increases only about one tenth of this. If you regulated your pendulum (not made of Invar) at one temperature, and the temperature increased by 1 degree Centigrade, the corresponding change in length is about $dl/l = 1 \times 10^{-5}$. Using Eq. (40), this corresponds to a fractional change in period of about 5×10^{-6}, or a gain of about three seconds per week. If the pendulum were made of Invar, this change would have been about one third of a second per week.

In a mercury-compensated pendulum, faster expanding mercury compensates for the slower expanding pendulum rod (typically of steel). A vial of mercury is seated on the tip of the rod, and as the rod expands downwards, the mercury expands upwards, so that the moment of inertia of the pendulum assembly stays at the same point as the temperature changes. To first order, the height of the mercury column, h_m, is related to the length of the rod, l_s, by:

$$l_s w_s = \tfrac{1}{2} h_m w_m \text{ ; } \tfrac{1}{2} \text{ because C of mass is at } \tfrac{1}{2} h_m \quad (42)$$

where w_m refers to the linear thermal coefficient of expansion for mercury, and w_s refers to steel. Using Eq. (42) for a steel pendulum with a period of two seconds, the height of mercury that compensates the expansion of steel is about ~~7.2~~ 14.3 inches. This is not precisely right, though, as pointed out in Sec V c in more detail.

A question arises as to how much the expansion of the glass vial's diameter reduces the rise of the mercury inside the vial. The area of the glass vial increases at a rate of twice the linear thermal coefficient of expansion of the glass, or about $1.7 \times 10^{-5} (^oC)^{-1}$. Since the glass vial is open ended, this is also the fractional increase in the vial's volume. The volume expansion of the mercury within the vial is $1.8 \times 10^{-4} (^oC)^{-1}$. The volume expansion of the glass vial reduces the mercury column by about one tenth, suggesting that the proper height of mercury for compensation should be about ten percent larger than the 7.2 inches, or 7.9 inches, but, as mentioned in the last paragraph, this is not precisely accurate (see Sec. V c).

V b. The Effect of Changing a Pendulum's Amplitude of Swing

The amplitude of swing, A, is influenced primarily by the design of the escapement, be it anchor, verge, or whatever. Because I wanted to focus on the physics of the simple pendulum, I don't include their effects here. Rawlings describes some of these effects in Ref. 1. We can, though, show how T is affected by variations in A without examining how the escapement produces a change in A. Using the infinite series expansion for T, Eq. (19), and taking the derivative of T with respect to A,

$$dT/dA = 2\pi(l/g)^{1/2}[\ 1/4 + (9/32)\ \sin^2(A/2) + (75/256)\ \sin^4(A/2) + \ldots]\ \sin(A/2)\ \cos(A/2) \qquad (43)$$

Figure 8 is a plot of dT/dA versus A. For example, suppose we have a pendulum with T = 1 sec. and A = 5 degrees. From Eq. (43) or Fig. 8, dT/dA is found to be 0.011. That is,

$$dT = 0.011\ dA \qquad (44)$$

If the amplitude of swing were to change by, say, one thousandth of a radian (about six hundredths of a degree), then the period would change by $dT = 1.1 \times 10^{-5}$ sec, enough to introduce an error in the clock's timekeeping of about seven seconds per week.

V c. The Effect of a Pendulum Rod's Mass.

Up to this point, the mass m of the pendulum rod was negligible. Although m is always considerably less than the mass M of the bob, it is never zero. When the mass of the rod is not zero, the equations of motion are changed. The moment of inertia of the dimensionless bob and the rod assembly about the suspension point is

$$I = Ml^2 + \int \mu r^2 dV = Ml^2 + ml^2/3 \qquad (45)$$

where μ is the density of the rod, r is the moment arm of the elemental volume dV, and l is the length of the rod.

The torque acts through the center of mass, whose distance from the suspension point

$$l\ (M + m/2)\ /\ (M + m) \qquad (46)$$

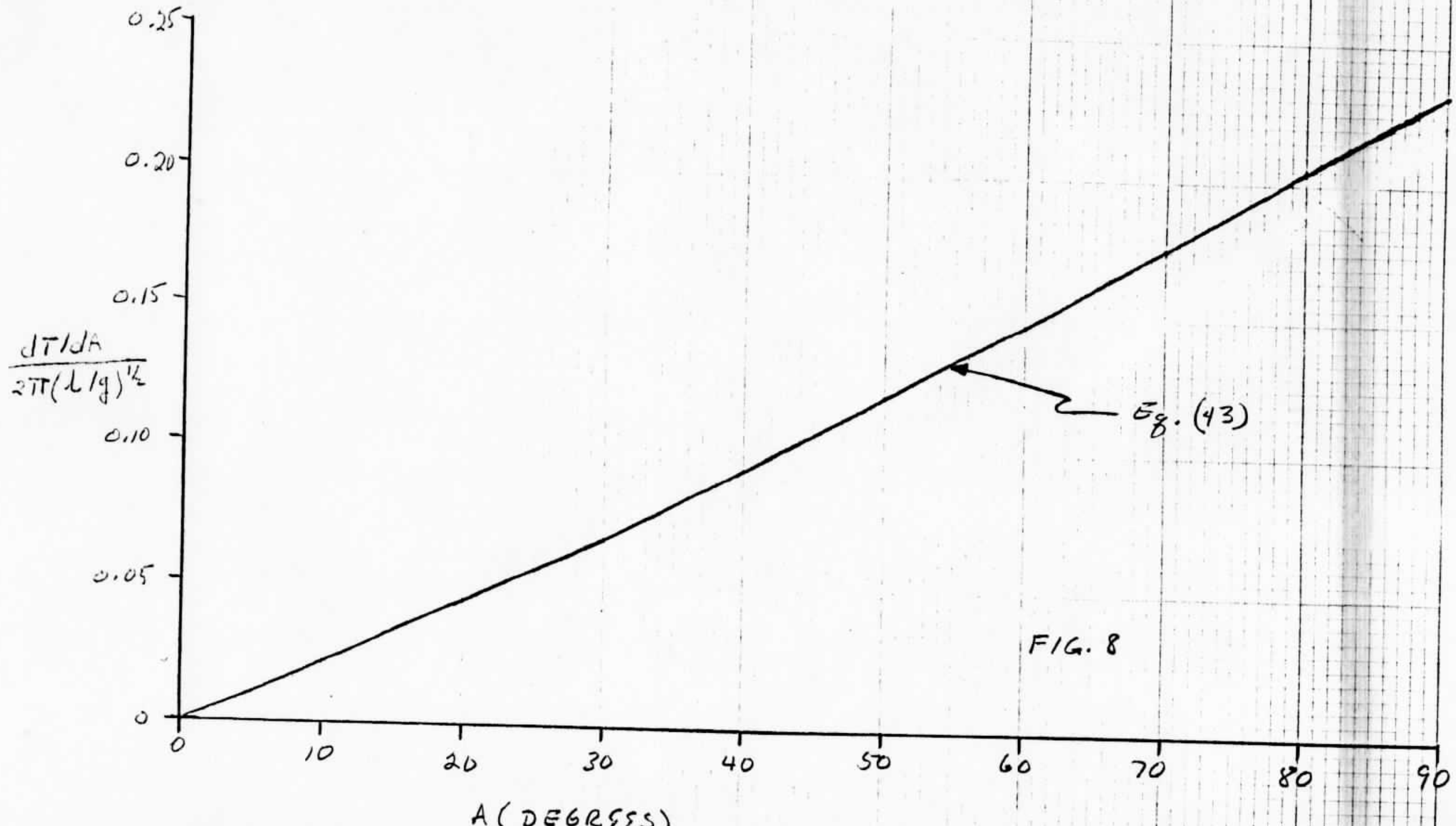

0.25
0.20
0.15
0.10
0.05
0
$\frac{dT/dA}{2\pi(l/g)^{1/2}}$
Eq. (43)
0
10
20
30
40
50
60
70
80
90
A (DEGREES)

FIG. 8

is the radius arm. With Eq. (45) for I, Eq. (46) for the radius arm, and M + m for the total mass of the assembly, Eq. (3) becomes

$$(M + m/3) l^2 d^2a/dt^2 = -(M + m)\ g l\ \frac{(M + m/2)}{(M + m)}\ \sin a \qquad (47)$$

This is an equation of motion that is identical to Eq. (5), but with l replaced by

$$l(M + m/3)/(M + m/2) \qquad (48)$$

This means that the equation of motion and the period of this rod and bob assembly are the same as those of the exact derivation in Section II, namely, Eq. (12) for a, and Eq. (16) for T. To account for the non-zero mass of the rod in those equations, though, l must be replaced by Eq. (48). With this replacement, the period becomes

$$T = 4(l/g)^{1/2}\ K[\sin^2(A/2)][(1 + m/3M)/(1 + m/2M)]^{1/2} \qquad (49)$$

Notice that the period of the pendulum depends on both M and m. This is different than the result found in Section II, where the period is independent of the mass of the bob. The reason why the period depends on M and m is because the rod and the bob have different shapes. With their different shapes, they contribute differently to the expression for the moment of inertia of the assembly, and they contribute still differently to the expression for the center of gravity of the assembly. These different contributions result in an equation of motion that does not simplify to the extent that it did in the derivation in Section II.

As an example of the error m can introduce in the timekeeping of a pendulum, consider a steel pendulum rod of 4 mm diameter and 1 m length with a mass of about 100 gm (the density of steel is about 7.8 gm/cc), a pendulum bob of mass M = 5 kg. Then the fractional error in the use of Eq. (2) for T instead of Eq. (47) for T is about 2×10^{-3}, representing a gain of about 2 1/3 minutes per day.

If m << M, then

$$(1 + m/3M)^{1/2} \simeq 1 + m/6M \qquad (50)$$

$$(1 + m/2M)^{-1/2} \simeq 1 - m/4M \qquad (51)$$

and the period, Eq. (49) becomes

$$T \simeq 4[l\ (1 - m/6M)/g]\ \ K[\sin^2(A/2)]\ ) \qquad (52)$$

In effect, l of Eq. (16) has been replaced by l (1 - m/6M).

The effect of the non-zero mass of the pendulum rod is to shorten the period. In effect, the center of mass of the bob and the rod is shifted upward from the center of the bob (where it would be if the mass of the rod were zero), so the period of the pendulum is the same as the period of a massless-rod pendulum of a shorter length.

In a likewise fashion, the shape of the bob (sphere, lens, cylinder, etc.) will alter the above equations for the moment of inertia of the pendulum assembly.

The relationship between the torque and the radial acceleration of the pendulum, Eq. (3), can be generalized by

$$- m_t g\, l_c \sin a = I\, d^2a/dt^2 \qquad (53)$$

where: m_t is the total mass of the pendulum assembly, both rod and bob; l_c is the radius arm, the distance from the suspension point to the center of mass of the assembly; and I is the moment of inertia of the assembly, taken about the suspension point. Rearranging Eq. (53) into the form of Eq. (5)

$$d^2a/dt^2 = - (g/l_c) \sin a \qquad (54)$$

where I define the true pendulum length to be

$$l_t = I \,/\, m_t l_c \qquad (55)$$

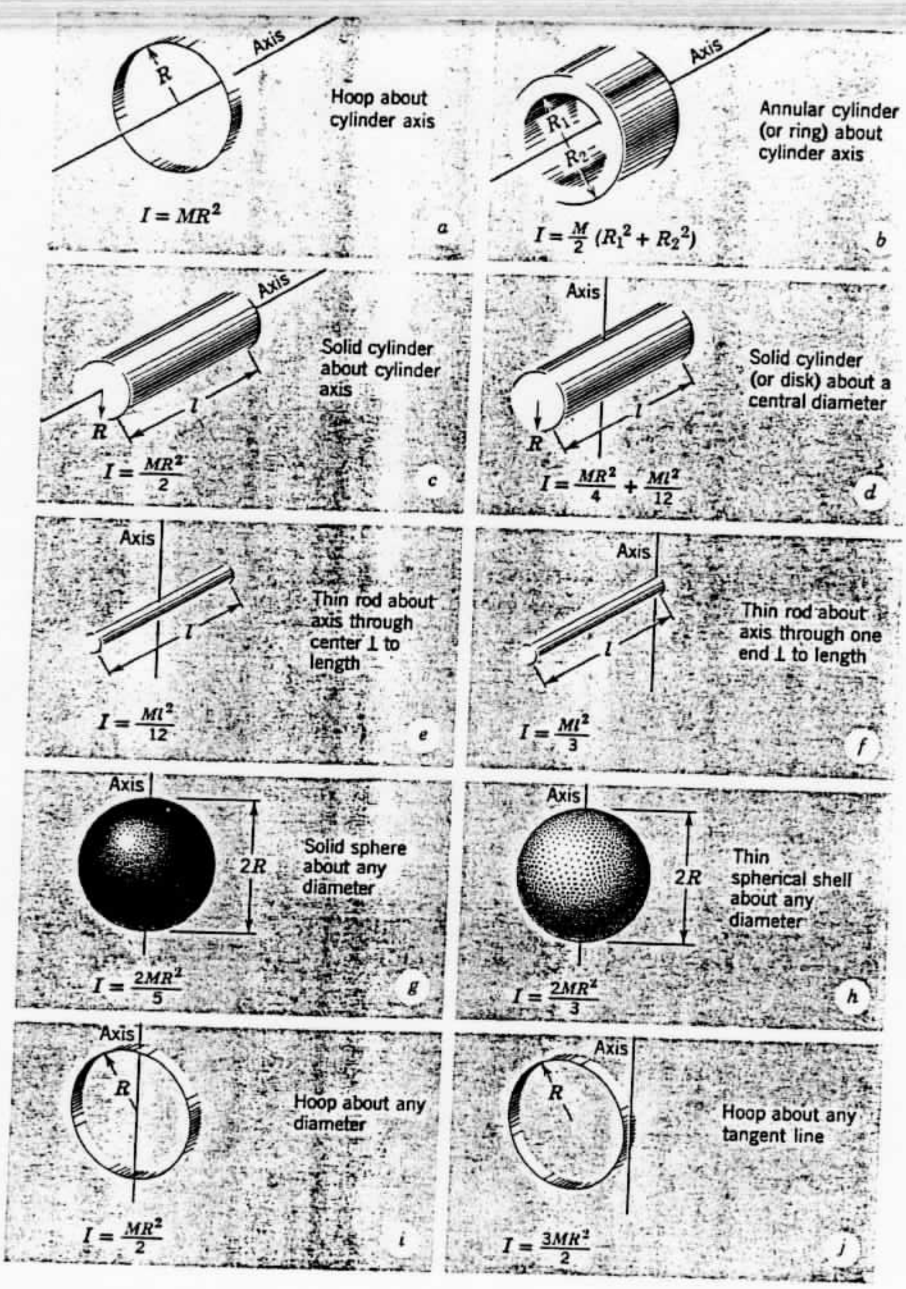
The exact solution to Eq. (54) is Eq. (12), but with l there replaced by l_t. The pendulum performs the same motion, but with a period that differs from Eq. (16), namely

$$T = 4\, (I \,/\, g\, m_t l_c)^{1/2}\, K[\, \sin^2(A/2)] \qquad (56)$$

Thus, the calculation of the true length of any pendulum assembly, and thereby its period, requires calculation of its center of mass and its moment of inertia.

Consider first the example of a spherical bob of radius R at the end of a massless rod, such that the center of the bob is a distance l from the suspension point. The moments of inertia of a variety of shapes are found in Ref. 8, and the relevant table is reproduced here as Fig. 9.

The moment of inertia of the bob about an axis through its center is $(2/5)mR^2$. Displacing that axis by l to the supsension point adds the moment ml^2. This displacement term comes via the Parallel Axis Theorem, which states that if the rotational axis is displaced by an amount l, and the axis



Axis
R
Hoop about cylinder axis
$I = MR^2$
a
Axis
R_1
R_2
Annular cylinder (or ring) about cylinder axis
$I = \frac{M}{2}(R_1^2 + R_2^2)$
b
Axis
R
l
Solid cylinder about cylinder axis
$I = \frac{MR^2}{2}$
c
Axis
R
l
Solid cylinder (or disk) about a central diameter
$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$
d
Axis
l
Thin rod about axis through center ⊥ to length
$I = \frac{Ml^2}{12}$
e
Axis
l
Thin rod about axis through one end ⊥ to length
$I = \frac{Ml^2}{3}$
f
Axis
2R
Solid sphere about any diameter
$I = \frac{2MR^2}{5}$
g
Axis
2R
Thin spherical shell about any diameter
$I = \frac{2MR^2}{3}$
h
Axis
R
Hoop about any diameter
$I = \frac{MR^2}{2}$
i
Axis
R
Hoop about any tangent line
$I = \frac{3MR^2}{2}$
j

remains parallel to the original axis, the resulting moment of inertia is the original moment of inertia plus ml^2. For the present example

$$I = (2/5)\ m\ R^2 + m\ l^2 \tag{57}$$

$$l_c = l \tag{58}$$

so that the true length of this pendulum assembly is

$$l_t = l\ (1 + 2R^2/5l^2) \tag{59}$$

The true length of this pendulum assembly is increased over that of the simple pendulum (with a bob of zero radius), so the pendulum in this example has a period longer than that of the simple pendulum. The ratio of the period of the pendulum in this example to that of the simple pendulum is

$$T\ (\text{this example})\ /\ T\ (\text{simple}) = (1 + 2R^2/5l^2)^{1/2} \tag{60}$$

Typically, $l >> R$, so

$$T\ (\text{this example})\ /\ T\ (\text{simple}) = 1 + R^2/5l^2 \tag{61}$$

A 5-kg steel pendulum bob (of density 7.8 gm/cc) with a radius of 4.67 cm at the end of a 1-m rod has a period of about 5.7×10^{-4} sec longer than that of a 1-m pendulum with a dimensionless bob. Although 5.7×10^{-4} sec doesn't seem like much, when compounded over a day's run time, it represents a loss of 49 seconds. In summary, the higher the density of the bob (hence the smaller the bob), the less the period is increased.

As an aside, what's the best metal to use? Lead? Gold? Platinum? Uranium? No, the two most dense metals are Osmium (22.5 gm/cc) and Indium (22.4 gm/cc). The density of Gold is 19.3 gm/cc, and Lead's density is a distant 11 gm/cc. At the other end of the density scale, Rubidium had a density of 1.53 gm/cc, Magnesium 1.74, Beryllium 1.84, Aluminum 2.7. Iron is in the middle at 7.9. If you made your pendulum bob out of Osmium instead of steel, the period of the pendulum would more closely approach that of the simple pendulum. By how much? Instead of your pendulum losing 49 seconds per day, with an Osmium bob, it would only lose 24 seconds per day.

A more complicated pendulum than the example presented above is a mercury-compensated pendulum, whose pendulum assembly consists of a long, thin pendulum rod of length l, made of steel, say, which is joined at its lower end to the bottom of a mercury column of radius R and height h. Intentionally, I neglect the mass of the container for the mercury. This pendulum assembly has a moment of inertia of

$$I = (1/3)\ m\ l^2 + (1/4)\ M\ R^2 + (1/12)\ M\ h^2 + M\ (l - h/2)^2 \qquad (62)$$

Here m is the mass of the rod, and M is the mass of the mercury column, or bob. The first term on the right is the contribution to I by the pendulum rod. The second and third are the contributions to I by the bob, referenced to a rotational axis at its center. The final term represents use of the parallel axis theorem to shift the rotational axis of the bob from the center of the mercury column to the suspension point. The radius arm is

$$l_c = [ml/2 + M\ (l - h/2)]\ /\ (m + M) \qquad (63)$$

The total mass is m + M. The true pendulum length simplifies to

$$l_t = l - h/2 + \frac{(1/6)h^2 + (1/2)R^2 + ml(h/2 - l/3)/M}{l(2 + m/M) - h} \qquad (64)$$

For this pendulum assembly to be compensated for thermal expansion, l_t must be independent of temperature. As can be seen from Eq. (64) and the appropriate versions of Eq. (41) for a mercury bob and a steel rod, this is a very complicated relationship.

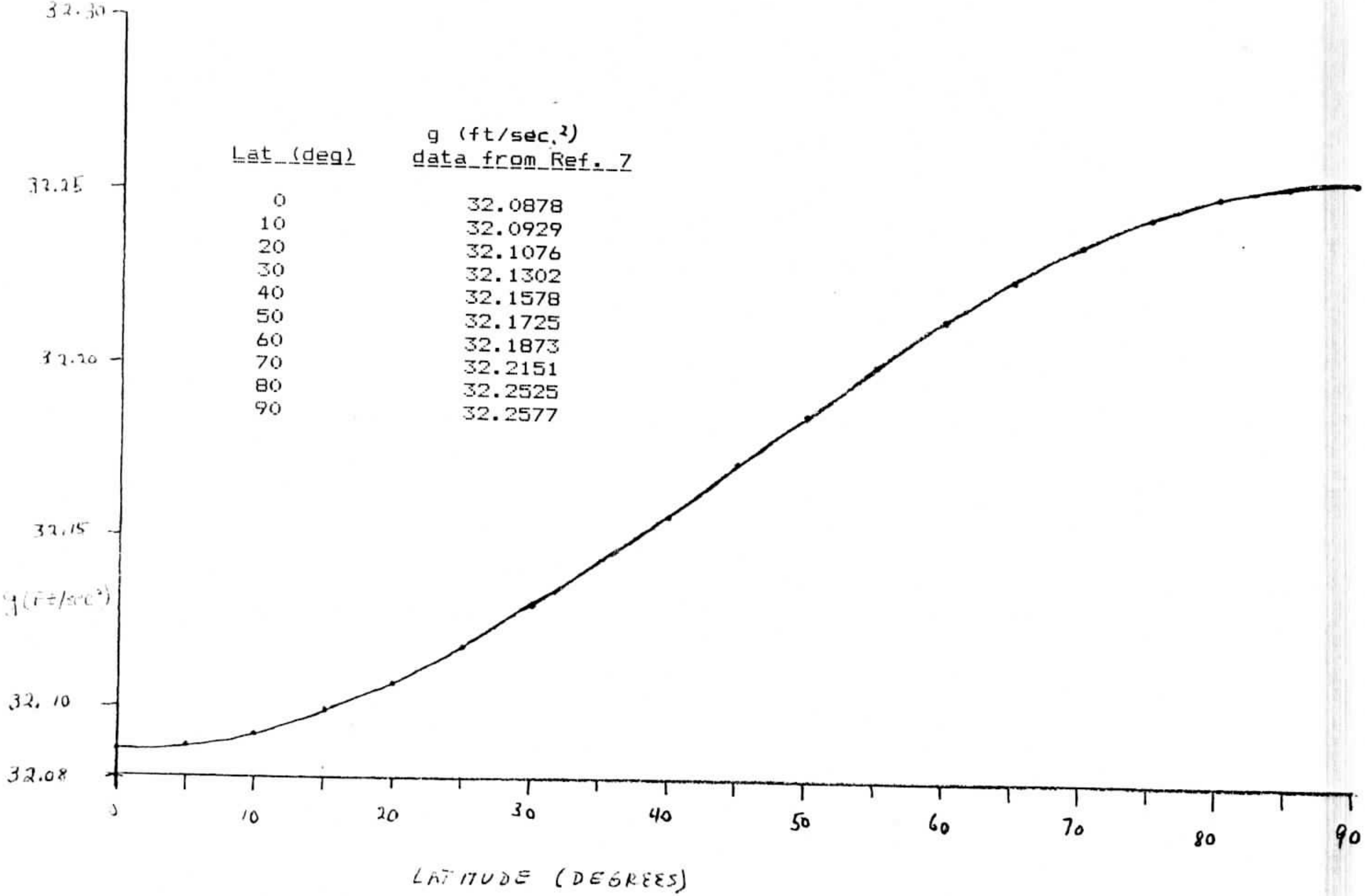
V d. The Effect of Changes in Gravity

The gravitational acceleration, g, is usually taken to be 32.2 ft/sec^2, but this varies from its smallest value, 32.0878 ft/sec^2 at the Earth's equator, to its greatest value, 32.2577 ft/sec^2 at the Poles. This dependence of g on latitude, be it north or south, is shown in Fig. 10 (which is based on data in Ref. 7). The value of g decreases with altitude above sea level, but not by much. This decrease is -3.086×10^{-6} ft/sec^2/ft (Ref. 7). For a 'standard reference', the value of g at Greenwich (latitude 51°28.6', elevation 157 ft.) is 32.1912 ft/sec^2.

Because the functional form of Eq. (38) involves g in much the same way as it involves l, the relation

$$dT/T = -\ (1/2)\ dg/g \qquad (65)$$

can be obtained in much the same way as was the variation of T with l. The minus sign means that T decreases as g



Lat (deg)	g (ft/sec.²) data from Ref. 7
0	32.0878
10	32.0929
20	32.1076
30	32.1302
40	32.1578
50	32.1725
60	32.1873
70	32.2151
80	32.2525
90	32.2577

FIG. 10

increases. If you set your pendulum clock correctly at Monrovia, Liberia (latitude 6°19', elevation 135 feet, where $g = 32.0920$ ft/sec^2) and move it to Karajak Glacier, Greenland (latitude 70°26.9', elevation 66 feet, where $g = 32.2354$ ft/sec^2), the period of your clock will decrease by 0.222 %, or 192 seconds per day.

Because the gravitational force is directed towards the center of the Earth, the direction of the gravitational force varies as the pendulum swings. Thus the force vectors at the ends of the swing of the pendulum aren't parallel, but are angled inwards ever so slightly. As pointed out by Rawlings in Ref. 1, the small-angle approximation to the period then takes the form

$$T = 2\pi(l/g)^{1/2}[R/(R+l)]^{1/2} \qquad (66)$$

where R is the Earth's radius (3960 mi.). The difference between using this equation and using Eq. (2) amounts to about 2 seconds per year.

One assumption we made in Secs. II and III is that g does not vary during the swing of the pendulum. The variation of g with height is very small, so the influence of this variation on the period of a pendulum is also small. For a pendulum that beats seconds with a 10 degree amplitude of swing, this change in g with height introduces an error that is less than six-hundredths of a second in the course of a year. The Earth's tides are manifestations of changes in g caused by changes in solar and lunar orientation. These changes do affect the motion of an accurate pendulum-driven regulator.

VI SUMMARY

The simple pendulum, consisting of a bob, supported by a massless and inextensible rod, moving under the force of gravity and no other force, is, of course, an idealization not realized in practice. Its physics, however, illustrates the factors that affect the motion of more complicated pendulum assemblies.

The period of the simple pendulum depends on its length l, the acceleration g due to gravity, and the amplitude A of swing. Changes in temperature produce changes in l; changes in latitude and elevation produce changes in g; and changes over time in frictional forces (not discussed here) in the going train and changes in barometric pressure (resistance to bob movement) produce changes in A. There are many other factors that produce changes in l, g, and A.

Is the simple pendulum isochronal? The answer is a conditional yes. The simple pendulum is isochronal to the extent that l, g, and A do not vary.

ACKNOWLEDGMENTS

In the course of developing this compendium, I have received suggested improvements from Pierre Boucheron, George Feinstein, Ernie Martt, and Snowden Taylor. To each of you, heartfelt thanks. I have incorporated your suggestions to the extent that I can. This compendium is improved as a result of your thoughtful help. Any errors or misrepresentations that remain are solely my responsibility. Hopefully, they are at a minimum.

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APPENDIX A

The tables of values of F(u\v) presented below for the Elliptic Integral of the First Kind are reproduced from similar tables presented in Ref. 3.

ELLIPTIC INTEGRAL OF THE FIRST KIND $F(\varphi\backslash\alpha)$

$$F(\varphi\backslash\alpha)=\int_0^{\varphi}(1-\sin^2\alpha\,\sin^2\theta)^{-\frac{1}{2}}\,d\theta$$

$\alpha\backslash\varphi$	0°	5°	10°	15°	20°	25°	30°
0°	0	0.08726 646	0.17453 293	0.26179 939	0.34906 585	0.43633 231	0.52359 878
2	0	0.08726 660	0.17453 400	0.26180 298	0.34907 428	0.43634 855	0.52362 636
4	0	0.08726 700	0.17453 721	0.26181 374	0.34909 952	0.43639 719	0.52370 903
6	0	0.08726 767	0.17454 255	0.26183 163	0.34914 148	0.43647 806	0.52384 653
8	0	0.08726 860	0.17454 999	0.26185 656	0.34919 998	0.43659 086	0.52403 839
10	0	0.08726 980	0.17455 949	0.26188 842	0.34927 479	0.43673 518	0.52428 402
12	0	0.08727 124	0.17457 102	0.26192 707	0.34936 558	0.43691 046	0.52458 259
14	0	0.08727 294	0.17458 451	0.26197 234	0.34947 200	0.43711 606	0.52493 314
16	0	0.08727 487	0.17459 991	0.26202 402	0.34959 358	0.43735 119	0.52533 449
18	0	0.08727 703	0.17461 714	0.26208 189	0.34972 983	0.43761 496	0.52578 529
20	0	0.08727 940	0.17463 611	0.26214 568	0.34988 016	0.43790 635	0.52628 399
22	0	0.08728 199	0.17465 675	0.26221 511	0.35004 395	0.43822 422	0.52682 887
24	0	0.08728 477	0.17467 895	0.26228 985	0.35022 048	0.43856 733	0.52741 799
26	0	0.08728 773	0.17470 261	0.26236 958	0.35040 901	0.43893 430	0.52804 924
28	0	0.08729 086	0.17472 762	0.26245 392	0.35060 870	0.43932 365	0.52872 029
30	0	0.08729 413	0.17475 386	0.26254 249	0.35081 868	0.43973 377	0.52942 863
32	0	0.08729 755	0.17478 119	0.26263 487	0.35103 803	0.44016 296	0.53017 153
34	0	0.08730 108	0.17480 950	0.26273 064	0.35126 576	0.44060 939	0.53094 608
36	0	0.08730 472	0.17483 864	0.26282 934	0.35150 083	0.44107 115	0.53174 916
38	0	0.08730 844	0.17486 848	0.26293 052	0.35174 218	0.44154 622	0.53257 745
40	0	0.08731 222	0.17489 887	0.26303 369	0.35198 869	0.44203 247	0.53342 745
42	0	0.08731 606	0.17492 967	0.26313 836	0.35223 920	0.44252 769	0.53429 546
44	0	0.08731 992	0.17496 073	0.26324 404	0.35249 254	0.44302 960	0.53517 761
46	0	0.08732 379	0.17499 189	0.26335 019	0.35274 748	0.44353 584	0.53606 986
48	0	0.08732 765	0.17502 300	0.26345 633	0.35300 280	0.44404 397	0.53696 798
50	0	0.08733 149	0.17505 392	0.26356 191	0.35325 724	0.44455 151	0.53786 765
52	0	0.08733 528	0.17508 448	0.26366 643	0.35350 955	0.44505 593	0.53876 438
54	0	0.08733 901	0.17511 455	0.26376 936	0.35375 845	0.44555 469	0.53965 358
56	0	0.08734 265	0.17514 397	0.26387 020	0.35400 269	0.44604 519	0.54053 059
58	0	0.08734 620	0.17517 260	0.26396 842	0.35424 101	0.44652 487	0.54139 069
60	0	0.08734 962	0.17520 029	0.26406 355	0.35447 217	0.44699 117	0.54222 911
62	0	0.08735 291	0.17522 690	0.26415 509	0.35469 497	0.44744 153	0.54304 111
64	0	0.08735 605	0.17525 232	0.26424 258	0.35490 823	0.44787 348	0.54382 197
66	0	0.08735 902	0.17527 640	0.26432 556	0.35511 081	0.44828 459	0.54456 704
68	0	0.08736 182	0.17529 903	0.26440 362	0.35530 160	0.44867 252	0.54527 182
70	0	0.08736 442	0.17532 010	0.26447 634	0.35547 959	0.44903 502	0.54593 192
72	0	0.08736 681	0.17533 949	0.26454 334	0.35564 377	0.44936 997	0.54654 316
74	0	0.08736 898	0.17535 712	0.26460 428	0.35579 326	0.44967 538	0.54710 162
76	0	0.08737 092	0.17537 289	0.26465 883	0.35592 721	0.44994 944	0.54760 364
78	0	0.08737 262	0.17538 672	0.26470 671	0.35604 488	0.45019 046	0.54804 587
80	0	0.08737 408	0.17539 854	0.26474 766	0.35614 560	0.45039 699	0.54842 535
82	0	0.08737 528	0.17540 830	0.26478 147	0.35622 881	0.45056 775	0.54873 947
84	0	0.08737 622	0.17541 594	0.26480 795	0.35629 402	0.45070 168	0.54898 608
86	0	0.08737 689	0.17542 143	0.26482 697	0.35634 086	0.45079 795	0.54916 348
88	0	0.08737 730	0.17542 473	0.26483 842	0.35636 908	0.45085 596	0.54927 042
90	0	0.08737 744	0.17542 583	0.26484 225	0.35637 851	0.45087 533	0.54930 614

ELLIPTIC INTEGRAL OF THE FIRST KIND $F(\varphi\backslash\alpha)$

$$F(\varphi\backslash\alpha)=\int_0^{\varphi}(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$$

$\alpha\backslash\varphi$	35°	40°	45°	50°	55°	60°
0°	0.61086 524	0.69813 170	0.78539 816	0.87266 463	0.95993 109	1.04719 755
2	0.61090 819	0.69819 436	0.78548 509	0.87278 045	0.96008 037	1.04738 465
4	0.61103 691	0.69838 220	0.78574 574	0.87312 784	0.96052 821	1.04794 603
6	0.61125 108	0.69869 484	0.78617 974	0.87370 649	0.96127 450	1.04888 194
8	0.61155 010	0.69913 161	0.78678 644	0.87451 593	0.96231 911	1.05019 278
10	0.61193 318	0.69969 159	0.78756 494	0.87555 545	0.96366 180	1.05187 911
12	0.61239 927	0.70037 358	0.78851 403	0.87682 412	0.96530 224	1.05394 160
14	0.61294 707	0.70117 608	0.78963 221	0.87832 076	0.96723 998	1.05638 099
16	0.61357 504	0.70209 730	0.79091 768	0.88004 389	0.96947 438	1.05919 813
18	0.61428 140	0.70313 511	0.79236 827	0.88199 174	0.97200 462	1.06239 384
20	0.61506 406	0.70428 706	0.79398 143	0.88416 214	0.97482 960	1.06596 891
22	0.61592 071	0.70555 037	0.79575 422	0.88655 254	0.97794 790	1.06992 405
24	0.61684 871	0.70692 183	0.79768 324	0.88915 992	0.98135 773	1.07425 976
26	0.61784 515	0.70839 788	0.79976 461	0.89198 071	0.98505 681	1.07897 628
28	0.61890 682	0.70997 451	0.80199 389	0.89501 076	0.98904 227	1.08407 347
30	0.62003 018	0.71164 728	0.80436 610	0.89824 524	0.99331 059	1.08955 067
32	0.62121 138	0.71341 124	0.80687 558	0.90167 852	0.99785 743	1.09540 656
34	0.62244 622	0.71526 098	0.80951 599	0.90530 415	1.00267 749	1.10163 899
36	0.62373 019	0.71719 052	0.81228 024	0.90911 465	1.00776 438	1.10824 474
38	0.62505 840	0.71919 335	0.81516 039	0.91310 148	1.01311 039	1.11521 933
40	0.62642 563	0.72126 235	0.81814 765	0.91725 487	1.01870 633	1.12255 667
42	0.62782 630	0.72338 982	0.82123 227	0.92156 370	1.02454 127	1.13024 880
44	0.62925 446	0.72556 741	0.82440 346	0.92601 535	1.03060 230	1.13828 546
46	0.63070 385	0.72778 615	0.82764 941	0.93059 558	1.03687 427	1.14665 369
48	0.63216 783	0.73003 640	0.83095 712	0.93528 835	1.04333 948	1.15533 731
50	0.63363 947	0.73230 789	0.83431 247	0.94007 568	1.04997 735	1.16431 637
52	0.63511 150	0.73458 970	0.83770 010	0.94493 756	1.05676 412	1.17356 652
54	0.63657 639	0.73687 028	0.84110 344	0.94985 177	1.06367 248	1.18305 833
56	0.63802 636	0.73913 751	0.84450 468	0.95479 381	1.07067 128	1.19275 650
58	0.63945 343	0.74137 870	0.84788 483	0.95973 682	1.07772 516	1.20261 907
60	0.64084 944	0.74358 071	0.85122 375	0.96465 156	1.08479 434	1.21259 661
62	0.64220 613	0.74572 998	0.85450 024	0.96950 647	1.09183 436	1.22263 139
64	0.64351 521	0.74781 266	0.85769 220	0.97426 773	1.09879 601	1.23265 660
66	0.64476 839	0.74981 471	0.86077 677	0.97889 946	1.10562 535	1.24259 576
68	0.64595 751	0.75172 208	0.86373 057	0.98336 406	1.11226 392	1.25236 238
70	0.64707 458	0.75352 078	0.86652 996	0.98762 253	1.11864 920	1.26185 988
72	0.64811 189	0.75519 716	0.86915 135	0.99163 507	1.12471 530	1.27098 218
74	0.64906 209	0.75673 800	0.87157 159	0.99536 166	1.13039 401	1.27961 482
76	0.64991 829	0.75813 076	0.87376 830	0.99876 287	1.13561 610	1.28763 696
78	0.65067 415	0.75936 376	0.87572 037	1.00180 067	1.14031 304	1.29492 436
80	0.65132 394	0.76042 640	0.87740 833	1.00443 942	1.14441 892	1.30135 321
82	0.65186 270	0.76130 931	0.87881 481	1.00664 678	1.14787 262	1.30680 495
84	0.65228 622	0.76200 457	0.87992 495	1.00839 470	1.15062 010	1.31117 166
86	0.65259 116	0.76250 582	0.88072 675	1.00966 028	1.15261 652	1.31436 170
88	0.65277 510	0.76280 846	0.88121 143	1.01042 658	1.15382 828	1.31630 510
90	0.65283 658	0.76290 965	0.88137 359	1.01068 319	1.15423 455	1.31695 790

ELLIPTIC INTEGRAL OF THE FIRST KIND $F(\varphi\backslash\alpha)$

$$F(\varphi\backslash\alpha)=\int_0^{\varphi}(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$$

$\alpha\backslash\varphi$	65°	70°	75°	80°	85°	90°
0°	1.13446 401	1.22173 048	1.30899 694	1.39626 340	1.48352 986	1.57079 633
2	1.13469 294	1.22200 477	1.30931 959	1.39663 672	1.48395 543	1.57127 495
4	1.13537 994	1.22282 810	1.31028 822	1.39775 763	1.48523 342	1.57271 244
6	1.13652 576	1.22420 180	1.31190 491	1.39962 909	1.48736 769	1.57511 361
8	1.13813 158	1.22612 810	1.31417 314	1.40225 598	1.49036 470	1.57848 658
10	1.14019 906	1.22861 010	1.31709 778	1.40564 522	1.49423 361	1.58284 280
12	1.14273 032	1.23165 180	1.32068 514	1.40980 577	1.49898 627	1.58819 721
14	1.14572 789	1.23525 808	1.32494 296	1.41474 871	1.50463 742	1.59456 834
16	1.14919 471	1.23943 470	1.32988 047	1.42048 728	1.51120 474	1.60197 853
18	1.15313 409	1.24418 827	1.33550 840	1.42703 700	1.51870 904	1.61045 415
20	1.15754 967	1.24952 627	1.34183 901	1.43441 578	1.52717 445	1.62002 590
22	1.16244 535	1.25545 700	1.34888 616	1.44264 399	1.53662 865	1.63072 910
24	1.16782 525	1.26198 957	1.35666 531	1.45174 466	1.54710 309	1.642o0 414
26	1.17369 362	1.26913 385	1.36519 359	1.46174 360	1.55863 334	1.65569 693
28	1.18005 472	1.27690 045	1.37448 981	1.47266 958	1.57125 942	1.67005 943
30	1.18691 274	1.28530 059	1.38457 455	1.48455 455	1.58502 624	1.68575 035
32	1.19427 162	1.29434 605	1.39547 013	1.49743 384	1.59998 406	1.70283 594
34	1.20213 489	1.30404 906	1.40720 064	1.51134 644	1.61618 906	1.72139 083
36	1.21050 542	1.31442 210	1.41979 198	1.52633 523	1.63370 398	1.74149 923
38	1.21938 520	1.32547 772	1.43327 179	1.54244 734	1.65259 894	1.76325 618
40	1.22877 499	1.33722 824	1.44766 938	1.55973 441	1.67295 226	1.78676 913
42	1.23867 392	1.34968 545	1.46301 565	1.57825 301	1.69485 156	1.81215 985
44	1.24907 904	1.36286 013	1.47934 287	1.59806 493	1.71839 498	1.83956 672
46	1.25998 475	1.37676 148	1.49668 437	1.61923 762	1.74369 264	1.86914 755
48	1.27138 210	1.39139 640	1.51507 416	1.64184 453	1.77086 836	1.90108 303
50	1.28325 798	1.40676 855	1.53454 619	1.66596 542	1.80006 176	1.93558 110
52	1.29559 414	1.42287 717	1.55513 354	1.69168 665	1.83143 068	1.97288 227
54	1.30836 604	1.43971 560	1.57686 709	1.71910 125	1.86515 414	2.01326 657
56	1.32154 149	1.45726 935	1.59977 378	1.74830 880	1.90143 591	2.05706 232
58	1.33507 910	1.47551 372	1.62387 409	1.77941 482	1.94050 873	2.10465 766
60	1.34892 643	1.49441 087	1.64917 867	1.81252 953	1.98263 957	2.15651 565
62	1.36301 803	1.51390 609	1.67568 359	1.84776 547	2.02813 570	2.21319 470
64	1.37727 323	1.53392 332	1.70336 398	1.88523 335	2.07735 219	2.27537 643
66	1.39159 384	1.55435 972	1.73216 516	1.92503 509	2.13070 052	2.34390 472
68	1.40586 195	1.57507 940	1.76199 085	1.96725 237	2.18865 839	2.41984 165
70	1.41993 796	1.59590 624	1.79268 736	2.01192 798	2.25177 995	2.50455 008
72	1.43365 925	1.61661 644	1.82402 292	2.05903 582	2.32070 416	2.59981 973
74	1.44684 001	1.63693 134	1.85566 175	2.10843 282	2.39615 610	2.70806 762
76	1.45927 266	1.65651 218	1.88713 308	2.15978 295	2.47892 739	2.83267 258
78	1.47073 163	1.67495 873	1.91779 814	2.21243 977	2.56980 281	2.97856 895
80	1.48098 006	1.69181 489	1.94682 231	2.26527 326	2.66935 045	3.15338 525
82	1.48977 975	1.70658 456	1.97316 666	2.31643 897	2.77736 748	3.36986 803
84	1.49690 410	1.71876 033	1.99562 118	2.36313 736	2.89146 664	3.65185 597
86	1.50215 336	1.72786 543	2.01290 452	2.40153 358	3.00370 926	4.05275 817
88	1.50537 033	1.73350 464	2.02384 126	2.42718 003	3.09448 898	4.74271 727
90	1.50645 424	1.73541 516	2.02758 942	2.43624 605	3.13130 133	∞

APPENDIX B

The tables of values of K(z) presented below for the Complete Elliptic Integral of the First Kind are reproduced from similar tables presented in Ref. 3.

COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

$$K(m)=\int_0^{\frac{\pi}{2}}(1-m\sin^2\theta)^{-\frac{1}{2}}d\theta$$

m	$K(m)$	m	$K(m)$
0.00	1.57079 63267 94897	1.00	∞
0.01	1.57474 55615 17356	0.99	3.69563 73629 89875
0.02	1.57873 99120 07773	0.98	3.35414 14456 99160
0.03	1.58278 03424 06373	0.97	3.15587 49478 91841
0.04	1.58686 78474 54166	0.96	3.01611 24924 77648
0.05	1.59100 34537 90792	0.95	2.90833 72484 44552
0.06	1.59518 82213 21610	0.94	2.82075 24967 55872
0.07	1.59942 32446 58510	0.93	2.74707 30040 24667
0.08	1.60370 96546 39253	0.92	2.68355 14063 15229
0.09	1.60804 86199 30513	0.91	2.62777 33320 84344
0.10	1.61244 13487 20219	0.90	2.57809 21133 48173
0.11	1.61688 90905 05203	0.89	2.53333 45460 02200
0.12	1.62139 31379 80658	0.88	2.49263 53232 39716
0.13	1.62595 48290 38433	0.87	2.45533 80283 21380
0.14	1.63057 55488 81754	0.86	2.42093 29603 44303
0.15	1.63525 67322 64580	0.85	2.38901 64863 25580
0.16	1.63999 98658 64511	0.84	2.35926 35547 45007
0.17	1.64480 64907 98881	0.83	2.33140 85677 50251
0.18	1.64967 82052 94514	0.82	2.30523 17368 77189
0.19	1.65461 66675 22527	0.81	2.28054 9138[illegible] 22770
0.20	1.65962 35986 10528	0.80	2.25720 53268 20854
0.21	1.66470 07858 45692	0.79	2.23506 77552 60349
0.22	1.66985 00860 83368	0.78	2.21402 24978 46332
0.23	1.67507 34293 77219	0.77	2.19397 09253 19189
0.24	1.68037 28228 48361	0.76	2.17482 70902 46414
0.25	1.68575 03548 12596	0.75	2.15651 56474 99643
0.26	1.69120 81991 86631	0.74	2.13897 01837 52114
0.27	1.69674 86201 96168	0.73	2.12213 18631 57396
0.28	1.70237 39774 10990	0.72	2.10594 83200 52758
0.29	1.70808 67311 34606	0.71	2.09037 27465 52360
0.30	1.71388 94481 78791	0.70	2.07536 31352 92469
0.31	1.71978 48080 56405	0.69	2.06088 16467 30131
0.32	1.72577 56096 29320	0.68	2.04689 40772 10577
0.33	1.73186 47782 52098	0.67	2.03336 94091 52233
0.34	1.73805 53734 56358	0.66	2.02027 94286 03592
0.35	1.74435 05972 25613	0.65	2.00759 83984 24376
0.36	1.75075 38029 15753	0.64	1.99530 27776 64729
0.37	1.75726 85048 82456	0.63	1.98337 09795 27821
0.38	1.76389 83888 83731	0.62	1.97178 31617 25656
0.39	1.77064 73233 33534	0.61	1.96052 10441 65830
0.40	1.77751 93714 91253	0.60	1.94956 77498 06026
0.41	1.78451 88046 81873	0.59	1.93890 76652 34220
0.42	1.79165 01166 52966	0.58	1.92852 63181 14418
0.43	1.79891 80391 87685	0.57	1.91841 02691 09912
0.44	1.80632 75591 07699	0.56	1.90854 70162 81211
0.45	1.81388 39368 16983	0.55	1.89892 49102 71554
0.46	1.82159 27265 56821	0.54	1.88953 30788 53096
0.47	1.82945 97985 64730	0.53	1.88036 13596 22178
0.48	1.83749 13633 55796	0.52	1.87140 02398 11034
0.49	1.84569 39983 74724	0.51	1.86264 08023 32739
0.50	1.85407 46773 01372	0.50	1.85407 46773 01372
		m	