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MATHEMATICS OF A WATCH ESCAPEMENT

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"This thou must always bear in mind, what is the nature of the whole….."
" Look always at the whole."
Marcus Aurelius, Philosopher and Emperor

1- Objective of this article

This article follows a similar one written in 2013 and published on Horological Science News, 2. 2013 (Mathematics of a clock escapement). The aim is here to offer a comprehensive and simple, yet useful and extensible, mathematical model of a watch escapement. I believe that a reasonably good mathematical model can be useful for watch design, modification and repair.
Many mathematical treatments of watch escapements specific issues are available [Defossez, Reymondin, Vermot, Rawlings, Daniels etc.] but I believe that an overall, versatile analytical model, based on the use of simple mathematics and of common computer applications (e.g. Spreadsheets and Vector Drawing Programs), together with the use of the mentioned already available specific studies, can still be useful and this is one of the objectives of this article. The Drawing Program necessary must allow the independent rotation and translation of the three escapement objects (Balance, Anchor and Wheel) in order to position them in the desired place for graphical derivations necessary to the calculation. As an alternative to the use of a Drawing Program, the method suggested in [Daniels] can be used. This method is based on the use of three scaled shapes for Wheel, Anchor and Balance, cut into a paper sheet, pinned on a drawing table or similar support, and moved individually in order to assume the desired positions with respect to the other parts of the escapement.
A number of experts in horological practice believe that the too many unknown watch features (friction losses in the first place) make almost useless any attempt to build a watch model. The presence of this difficulty, however, exists in any engineering undertaking, yet mathematical models are used everywhere. Examples are so easy to find that it is not necessary to list them here; I will only mention the model used (when Finite Element Models are not employed) to study stresses and deformations in a beam system, the De Saint Venant Solid Model, which is very far from the real thing to study. The key to the use of engineering mathematical models is to do the best in creating them (although imperfect), to test them against the real behaviour and to refine them, maybe using some empirical factor here and there, in order to reach a sufficient level of accuracy in the results.

2- Type of escapement chosen

The watch movement used for this exercise on model description is the one based on the Swiss Anchor Escapement shown in Fig. 1, which is also a reference for some nomenclature used.
Many descriptions are available on the functioning of this very ingenious movement [e.g. Defossez, Vermot] so they will not be replicated here; however, in the Appendix, a modified summary of the Defossez and Vermot descriptions is presented.
Many numerical data used are taken, for the sake of a consistent set, from [Vermot].
Symbols used and numerical data are listed in Section 10.
The here suggested calculation procedure could be applied with pertinent modifications, to other types of watch escapement mechanisms.

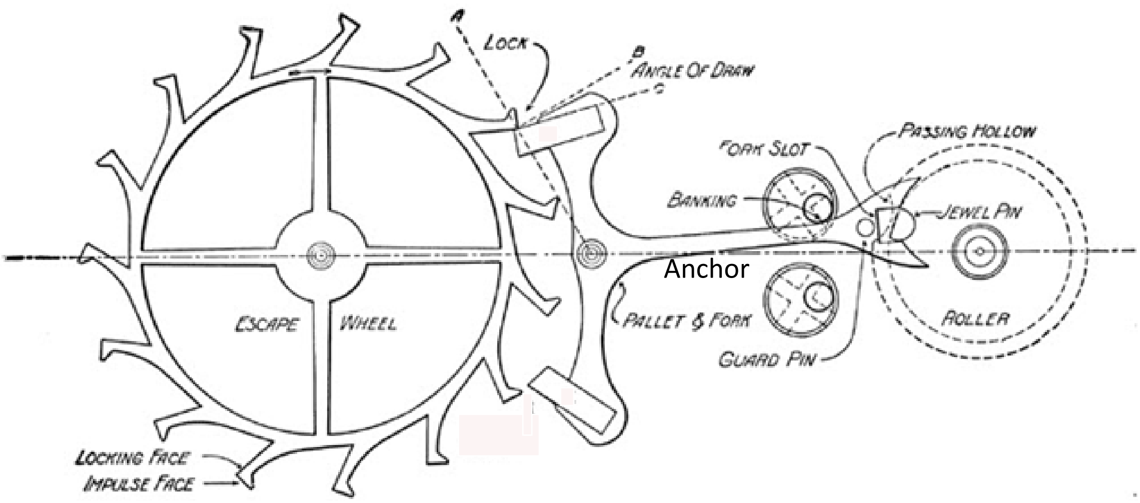

Figure 1, Watch lever escapement

3- Method used

The method proposed here consists in arbitrarily, yet with justification, splitting the mathematical model into two parts:

- The first part studies the movement of the three main parts of the escapement (Balance, Anchor and Escape Wheel) neglecting any friction loss and also the impulses received by the Balance from the Anchor. This first part can be named “basic model”.
- The second part studies energy losses (due to friction and other main effects) and energy gains (due ultimately to the action of the power spring) of the balance, as small perturbations of the basic model.

This procedure is justified by the very small entity of the Balance energy losses and gains as compared with the energy (kinetic plus spiral spring stored energy) of the Balance itself; the results of the "basic model" give a good approximation of the escapement motion, while the study of energy gains and losses of the Balance (second part of the model) serves to refine the overall result and, in particular, to determine the exact value of the amplitude of the Balance oscillation. An approximate value of this amplitude has to be initially "assumed" in the "basic model" on the basis of normally used values in the watch design practice.
Six decimal places have been used, where possible, through the calculations.
This method, I believe, makes the treatment of the problem rather simple and versatile, yet sufficiently accurate for practical uses.

4- Balance movement

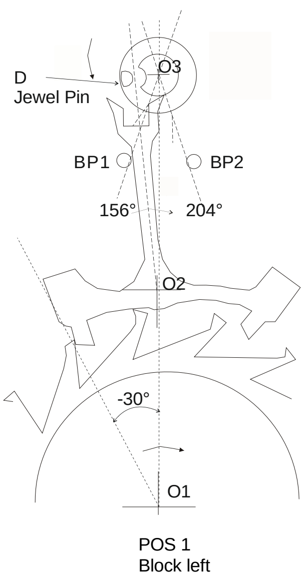

Fig. 2 Position 1 of the escapement
(Starting position)

Under the assumptions listed in Section 3, the equation of the Balance movement is, simply:

$$\ddot{\theta} + \omega_0^2 \; \theta = 0 \qquad (1)$$

Where:

$\boldsymbol{\vartheta}$ is the Balance angle measured counter-clockwise from $\overrightarrow{O3\ O1}$

ω_0 is the own rotational speed of the Balance $(= 2\pi/T)$, $15.708\ rad/s$

T is the own oscillation period of the Balance (=$2\pi\sqrt{J_b/K}$), 0.4 [s]

The solutions of equation (1) for θ and $\dot{\theta}$ are:

$$\theta=\theta_0 \sin(\omega_0 t) \qquad (2)$$

$$\dot{\theta}= \omega_0\, \theta_0 \cos(\omega_0 t) \qquad (3)$$

Where:

θ_0 is the amplitude of the oscillation of the Balance which is initially assumed equal to 270° (= 3/2 π [rad] = 4.7124 [rad] (a second iteration of the calculation on the basis of the energy balance evaluation, will suggest the need to refine this initial value).

t is the time [s]

The oscillation, then, spans from +270° to -270° and vice-versa. This angle is usually kept at a rather high value in order to keep high the energy of the Balance (and so to ease the movement imposed to the Anchor fork)and to minimize the effect of accidental shocks on the own period of oscillation of the Balance itself as shown by the Airy's formula for short shocks (instantaneous variation of the rotational speed) [Defossez]:

$$\Delta T = \frac{J_b}{K}\frac{\vartheta\ d\dot{\vartheta}}{\vartheta_0^2} \qquad (4)$$

The spreadsheet results for equations (2) and (3) are shown in Fig. 3. The first and the second columns show the time t[s] with two different starting points: in the first column the time runs from 0 at the position θ=0 of the balance, while in the second column the time assumes the 0 value at the impact of the Jewel Pin D against the Anchor fork towards right. The second column is the one used to draw the graph of the Balance oscillation. In Fig. 3, the drawing containing two circular arrow segments represents the sequence and amplitudes of the various phases of the Balance movement.

Points 1 and 3 on the graph indicate the instants of left and right impact of the Balance Jewel Pin with the Anchor fork; points 2 and 4 indicate the instants when the fork is left by the Jewel Pin near , respectively, the right and left Banking Pins (BP2 and 1).

The graph in Fig. 3 would be very slightly altered (and in a measure which could not be drawn in one sheet of normal paper) if the impulses originated by the Wheel and Anchor were considered. Also recalling a specific calculation of Balance- Anchor fork shocks (5 shocks considered) [Vermot, CD Section “Dégagement d’entrée”] the variation (before and after each shock) of the rotational speed of the Balance is considered zero for a calculation with two decimal places. This means that the maximum variation of the same rotational speed for each shock is lower than 0.01 [s^{-1}]. Then, according to the Airy’s formula (4), the variation of Balance own period at each shock is lower than ΔT_{var}:

$$\Delta T_{var} = \frac{J_b}{K}\frac{\vartheta\, d\dot{\vartheta}}{\vartheta_0^2} \;=\; \frac{2*10^{-6}}{5.03*10^{-4}} * \frac{24°}{57.2958} * \frac{0.01}{4.7124^2} = 7.5 * 10^{-7}\ [\mathrm{s}] \qquad (5)$$

	A	B	C	D	E	F	G
1	t[s]	t[s]fin	teta[rad]	t[s]	t[s]fin	teta °	teta'
2	-0,005665	0,000000	-0,41878	-0,005665	0,000000	-23,994476	73,729500
3	0,000000	0,005665	0	0,000000	0,005665	0	74,022379
4	0,005665	0,01133	0,418782	0,005665	0,011330	23,994476	73,729500
5	0,020000	0,025665	1,456211	0,020000	0,025665	83,434777	70,399449
6	0,040000	0,045665	2,769877	0,040000	0,045665	158,702339	59,885299
7	0,060000	0,065665	3,812408	0,060000	0,065665	218,434938	43,509131
8	0,080000	0,085665	4,481751	0,080000	0,085665	256,785505	22,873966
9	0,100000	0,105665	4,712387	0,100000	0,105665	270,000000	-0,000272
10	0,120000	0,125665	4,481740	0,120000	0,125665	256,784892	-22,874483
11	0,140000	0,145665	3,812387	0,140000	0,145665	218,433772	-43,509571
12	0,160000	0,165665	2,769849	0,160000	0,165665	158,700734	-59,885618
13	0,180000	0,185665	1,456178	0,180000	0,185665	83,432891	-70,399617
14	0,194330	0,199995	0,419117	0,194330	0,199995	24,013622	-73,729032
15	0,200000	0,205665	-0,000035	0,200000	0,205665	-0,001984	-74,022379
16	0,205668	0,211333	-0,419016	0,205668	0,211333	-24,007857	-73,729173
17	0,220000	0,225665	-1,456244	0,220000	0,225665	-83,436664	-70,399281
18	0,240000	0,245665	-2,769905	0,240000	0,245665	-158,703944	-59,884979
19	0,260000	0,265665	-3,812428	0,260000	0,265665	-218,436104	-43,508691
20	0,280000	0,285665	-4,481762	0,280000	0,285665	-256,786118	-22,873449
21	0,300000	0,305665	-4,712387	0,300000	0,305665	-270,000000	0,000816
22	0,320000	0,325665	-4,481730	0,320000	0,325665	-256,784279	22,875001
23	0,340000	0,345665	-3,812367	0,340000	0,345665	-218,432606	43,510011
24	0,360000	0,365665	-2,769821	0,360000	0,365665	-158,699130	59,885938
25	0,380000	0,385665	-1,456145	0,380000	0,385665	-83,431004	70,399785
26	0,400000	0,405665	0,000069	0,400000	0,405665	0,003967	74,022379
27							

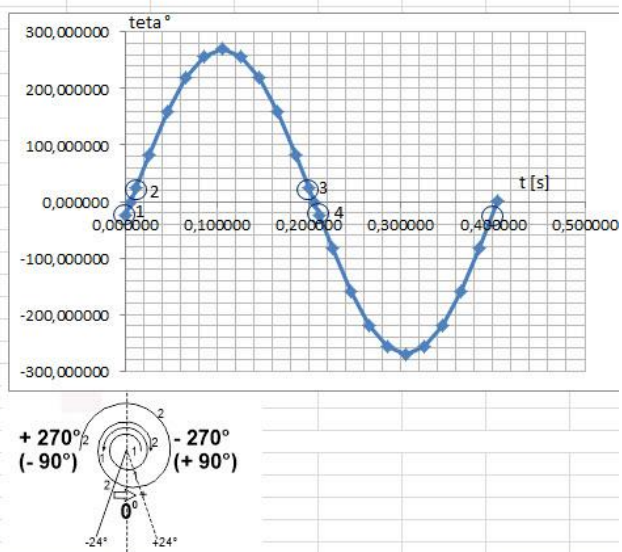

Fig. 3 – Balance movement as calculated in the “basic model”

The results (spreadsheet) of this section are one of the basis for calculation of the Wheel and of the Anchor movement in the next Sections.

5- Wheel movement

The Wheel angle of rotation α from the left block position (Fig. 1) to the right block position (Figure 10 in the APPENDIX, right) is composed of four phases [Vermot]:

1- Unlocking of the Wheel from the anchor left pallet (total angle from center point O1 =2 δ_e= 2*0.00582 [rad]=0.01164 [rad]= 2*0.33346°, value taken from a good drawing of the escapement); this angle corresponds to 0.0436 [rad] (2.5°) for the anchor (again from drawing, center O_2) and to 0.20944 [rad] (12°) for the Balance (center O_3).
2- First part of the left impulse to the anchor pallet (pallet impulse angle for Wheel α_p = 0.11345 [rad] = 6.5°, from drawing), corresponding to 0.10472= 6° for Anchor and to 0.4189 [rad] (24°)for Balance
3- Second part of the left impulse (tooth impulse angle for Wheel α_t = 0.06981[rad]= 4° from drawing), corresponding to 0.0436 [rad] = 2.5° for Anchor and to 0.1745[rad]=10° for Balance
4- Fall of the Wheel on the right pallet angle, α_f = 0.02618 [rad] =1.5°, corresponding to 0.00873 [rad] =0.5° for the Anchor

The first part of the left impulse starts at the end of unlocking and lasts until the "beak" of the Wheel tooth is in contact with the impulse plane face of the left pallet. After this point, the contact is between the plane face of the tooth and the plane face of the pallet (start of the second part of the left impulse). This passage [Vermot, 9.4] is marked by a significant drop in the couple transmitted to the Anchor by the Wheel, due to the mechanical differences of the two situations. This is the reason why this point is a likely position for a stop of the watch, if the power couple is too low. The contact ends when the plane face of the tooth comes in contact with the "beak" of the left pallet. The fall of the Wheel follows.

The times of rotation for these four angles are:

- Unlocking:

$$\frac{\delta_e}{\omega_w} = \frac{2*0.00582}{(3.8154+4.5772)/2} = \frac{2*0.00582}{4.1962} = 2*0.001387 = 0.002774\ s \quad (6)$$

Where : 3.8154 [rad/s] is the angular speed of the Wheel at the start of the unlocking (just after the first contact between Balance jewel pin and anchor fork, [Vermot pg. 565 and Section "Dégagement d'entrée" on enclosed CD] and 4.5772 [rad/s] is the corresponding value at the end of the unlocking. The average value between these two speeds is used here due to the small duration of the unlocking.
At the start (and, practically, during the unlocking), the rotational speed of the anchor is equal to 28.618 [rad/s] [Vermot, ibid.] and the rotational speed of the Balance is 73.73 [rad/s] (Fig.3). The value for the Anchor is obtained by the conservation of momentum in the impact between Balance and Anchor (see mass moments of inertia in the Section "Symbols used and Data").
The fact that the angular speed of the Wheel is so lower than the speed of the Anchor is explained by the fact that the two rotation circumferences are far from tangent to each other and intersect at an angle dependent on the initial angle of the Wheel (- α_0), on the Draw angle of the left Anchor pallet and on the unlocking angle ε as seen from the Anchor center of rotation O_2.
Once the rotation speed of the Anchor at the unlocking(= 28.618 [rad/s]) is known by the impact equation, the Wheel speeds can also be determined graphically on a good escapement drawing.

- First and second part of the impulse:

First part: $\frac{0.11345}{19.6615} = 0.005770$ [s]

Second part: $\frac{0.0698}{15.9947} = 0.0044$ [s] (7)

- Fall of the Wheel on the right pallet:
  The time taken in this part of the Wheel movement can be calculated by the equation governing the free motion of the Wheel under the action of the couple transmitted to it by the Gear Train C_w [N/mm]:

$$J_{gt} \frac{d\omega_w}{dt} = C_w \quad (8)$$

$$\omega_w = \frac{C_w}{J_{gt}} * t + \omega_{w0} \quad (9)$$

$$\propto_w = \omega_w * t = 9.048 * 10^4 * \frac{t^2}{2} + 15.9947 * t + \alpha_0 \quad (10)$$

α_0 is put to 0 (origin where the fall starts)

The positive solution of equation (10) is:

t = 0.0009578 [s] (11)

The total travel time of the Wheel from the left locked position to the fall on the right pallet, ($t_{w,\,l\text{-}r}$), is, then:

$t_{w,l\text{-}r}$= 0.002774 + 0.005770 + 0.0044 + 0.0009578 = 0.0139 [s] (12)

This time is equal to one half the travel time of the Wheel in one Balance period T (= 0.4 s).

In terms of average Wheel rotational speed when it is not stopped, having assumed a Wheel with 15 teeth (angle between two adjacent teeth = 360°/15=24°) the figure of (12), corrected for the back-and-forth unlocking movement of the Wheel, gives:

$$\omega_{w\ average\ in\ t_{w,l-r}} = \frac{12°}{(0.0139-0.002774)} = \frac{12°}{0.011126} = 1090\ °/s \quad (13)$$

The overall (including both stop times and moving times) average rotational speed of the Wheel is for comparison:

$$\omega_{w\ average\ in\ T} = \frac{2*12°}{T} = \frac{24°}{0.4} = 60°/s \qquad (14)$$

The Gear Train decreases this value by a factor of 10 for seconds hand (6°/s = 360°/minute), by 600 for minutes hand (0.1°/s=360°/hr) and by 7200 (0.0083333°/s=30°/hr).

The movement of the Wheel, according to the preceding data, is shown in Fig.4.

t(s)	alfa	
0	-30	lock left
0,0014	-30,3335	recoil
0,002774	-30	
0,008544	-23,5	
0,011544	-19,5	
0,0139	-18	lock right
0,199995	-18	
0,201395	-18,3335	recoil
0,202765	-18	
0,208535	-11,5	
0,211535	-7,5	
0,212493	-6	lock left
0,4	-6	end period

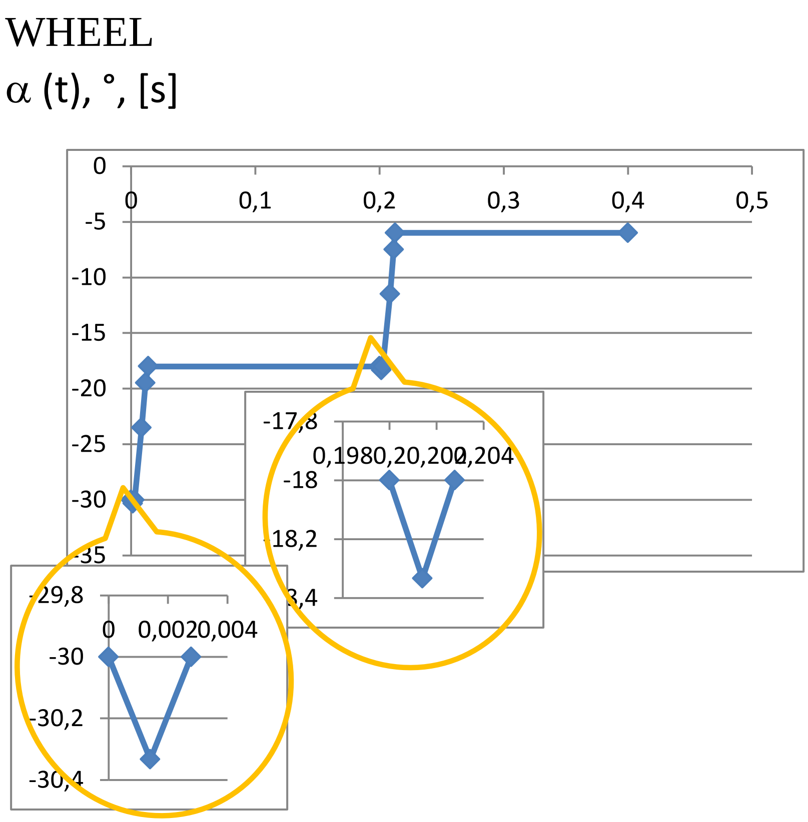

Fig. 4 – Wheel movement in one period

6- Anchor movement

From the previous Section 4, Balance movement and Section 5, Anchor movement, Fig. 5 can be drawn, which shows the anchor movement.
Fig.6 shows the time history of Jewel Pin – Anchor shocks after the first contact towards right [Vermot CD,Section “Dégagement d’entrée”]. The total duration of the shocks is so short that it cannot be shown in Fig.5 “Anchor movement”, where the shocks time and resulting anchor angles are, however, listed.
Fif. 7 shows the Balance, Anchor and Wheel movements together.

t [s]	angle [deg]
0,000000	6,000000
0,001090	5,999456
0,001091	5,998911
0,001680	5,998113
0,001681	5,997315
0,002070	5,996366
0,002071	5,995417
0,002320	5,994380
0,002321	5,993342
0,011330	-6,000000
0,200005	-6,000000
0,211333	6,000000
0,400000	6,000000

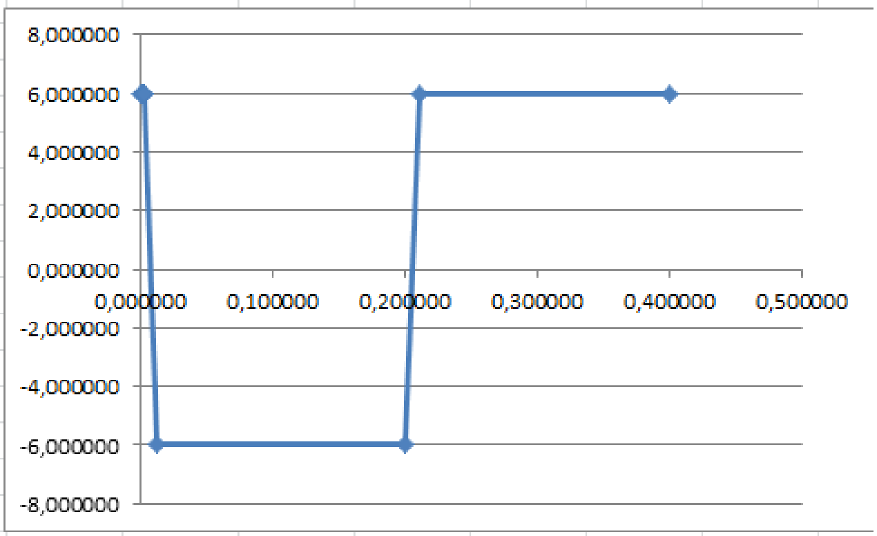

Fig. 5 Anchor movement

t(s)		r/s
0,294330	0,000000	28,618000
0,295420	0,001090	6,070000
0,295421	0,001091	24,672000
0,296010	0,001680	10,016000
0,296011	0,001681	22,107000
0,296400	0,002070	12,581000
0,296401	0,002071	20,440000
0,296650	0,002320	14,248000
0,296651	0,002321	19,357000

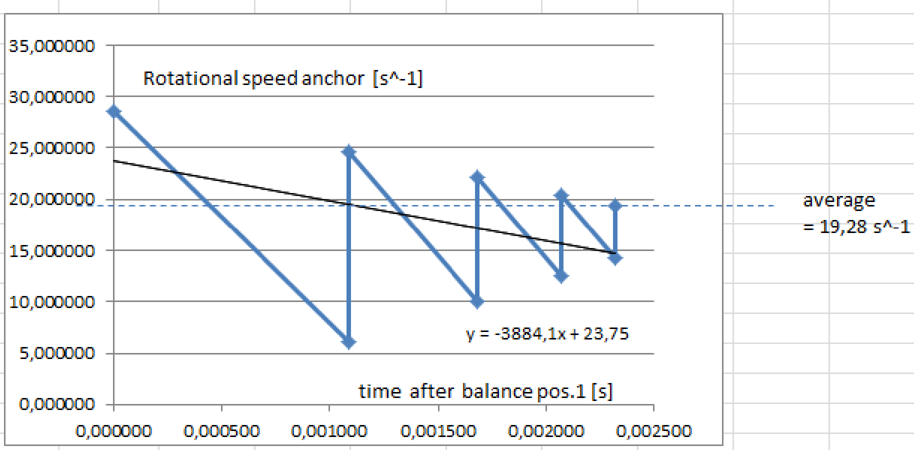

Fig. 6 Jewel Pin – Anchor shocks just after fork engagement towards right

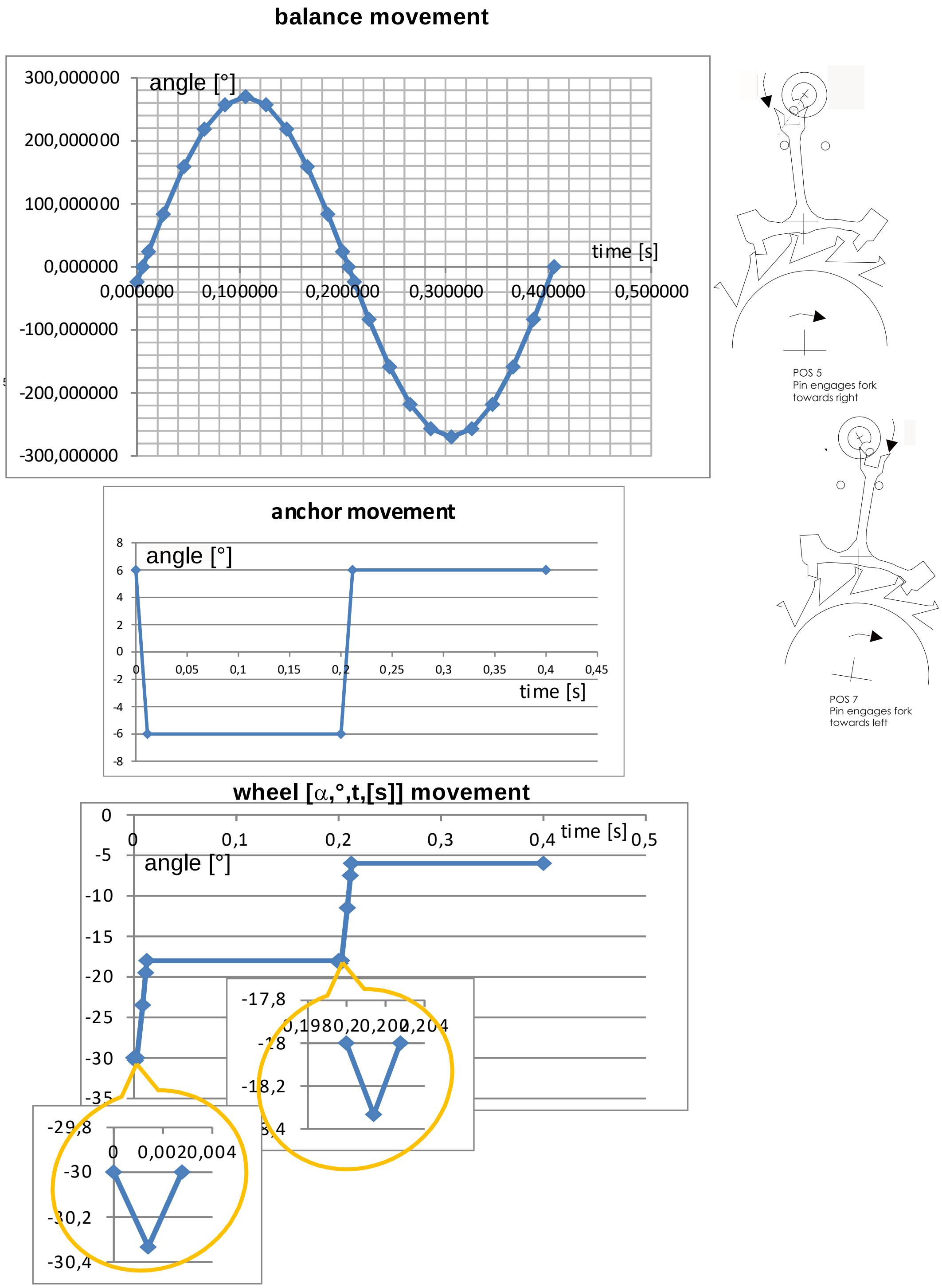

Fig.7 Balance, Anchor and Wheel movements shown together

7- Balance Energy considerations

7.1-Energy of the Balance

This quantity is equal to the maximum Kinetic energy of the Balance at the maximum amplitude angle (= 270°):

$$E_{cbmax} = \frac{1}{2} J_b \, \omega_{bmax}^2 = 0.5 * 2\ 10^{-6}\ 74.0224^2 =$$

$$= 5 * 10^{-1} * 2 * 10^{-6} * 5.48 * 10^{3} = 54.8 * 10^{-4} = 5.48 * 10^{-3} \text{ [N mm]} \qquad (15)$$

This quantity is calculated here as a reference for energy losses and gains of the Balance.

7.2 – Energy lost during left unlocking ($E_{l,\,lu}$)

$$E_{l,lu} = F_t * f * lock = \frac{C_w}{R_1} * f * lock = \frac{2.287 * 10^{-3}}{2.728} * 0.2 * 0.0436 = \frac{2.287}{2.728} * 10^{-3} * 2 * 10^{-1} * 4.36 * 10^{-2} = 7.310^{-6} [N\ mm] \qquad (16)$$

Where:

F_t is the tangential force transmitted by the Wheel to the Anchor pallet during unlocking, R1 is the contact radius of the Wheel .

f, is the friction coefficient between Wheel tooth and Anchor pallet (assumed equal to 0.2)

lock is the sliding path of the Wheel tooth on the Anchor pallet during unlocking (=0.0436 [mm], corresponding to ε equal to 2.5°) (see Section 5).
The final figure in (16) means that, because of symmetry between left and right lock, the energy lost by friction during one Balance period (0.4 [s]) is equal to

$$2*7.3\ 10^{-6}=1.46*10^{-5}\ [\text{N mm}] = 2.7\ \%\ \text{of the energy of the Balance.}$$

In the preceding calculation the energy lost by friction on the Anchor staff during unlocking is neglected because of its small relative amount.

7.3 Energy lost by the Balance during the initial shocks against the Anchor fork during unlocking

This energy, due to the small entity of the Anchor moment of inertia (≅ 0.5% of the Balance moment), is usually neglected (see Section 4- and equation (5). In any case, this energy can be estimated since it should have the order of magnitude (except for the influence of the restitution in shocks) to the Anchor kinetic energy after the first shocks of the Balance Jewel Pin and the Anchor fork.
The moment of inertia of the Anchor is equal to $1.02\ 10^{-8}$ [N mm s^2] (mass) and the rotational speed of the Anchor at the end of the shocks is 19.357 [s^{-1}] ([Vermot CD,Section "Dégagement d'entrée"]).
The Anchor energy after the shocks is, then:

$$E_{a,\,as}=\frac{1}{2}*1.02\ 10^{-8}*19.357^2=5*10^{-1}*1.02*10^{-8}*3.75*10^2=19.1*10^{-7}=1.91*$$
$$*\ 10^{-6}\ [N\ mm] \qquad (17)$$

The same amount of energy is also lost at the shocks after the right unlocking and, therefore, in one period

$E_{\text{lost after shocks}}=2*1.91*10^{-6}= 3.82 * 10^{-6}$ [N mm]

And, as a percentage of maximum Balance energy (15)

$3.82*10^{-6} / 5.4793\ 10^{-3} \cong 0.07\ \%$

This lost energy is really negligible in an overall balance.

7.4- Energy loss due to lateral friction in the balance (and in the Anchor) staff

It is assumed that the friction coefficient at this locations is equal to 0.15 (near the high end of the usual range 0.1 – 0.2 [Defossez]). The prevailing loss of energy is the one connected with the friction

between staff and the two bushings for forces on the staff perpendicular to its axis. The staff radius is assumed equal to 0.1 [mm]. Although the interested staffs are two (Balance and Anchor), here the reasoning is developed for one of them only, which represents both.
The lateral forces acting on the Balance staff are the force originated from the push of the Wheel on the Anchor pallet and (in case of a vertical position of the watch) the weight of the Balance. This force propagates to Anchor and Balance in a number of couples (which originate or perturb the motion of Anchor and Balance) and in two reaction forces from the Balance and Anchor. As just mentioned, these two reaction forces are unified for similarity of phenomena in this rather simple treatment, which, however, shows the interplay of the main parameters and gives a useful numerical result.
The tangential (peripheral) force F_t is equal to the one calculated in 7.2:

$$F_t = \frac{C_W}{R_1} = \frac{2.287*10^{-3}}{2.728} = 8.38*10^{-4} \text{ [N]} \qquad (18)$$

Adding the weight of the balance (vertical position of watch):

W_b =59.5 [mg] = 5.95 10^{-4} [N]

The energy lost for each alternance of the Balance is, then:

$(F_t + 5.95\ 10^{-4}) * f *0.1* \ \theta_0 = (8.38*10^{-4}+ 5.95\ 10^{-4}) * 0.15 *0.1 * 4.7124 = 1.433*10^{-3}*1.5*10^{-1}*1\ 10^{-1}*4.7124= 10.13*10^{-5}= 1.013\ \ 10^{-4}$ [N mm]

And for one period of Balance oscillation:

$$E_{l,f} = 2.03\ 10^{-4} \text{ [N mm]} \qquad (19)$$

The above presented evaluation can be refined at will dividing the run of the Anchor in various phases (lock, impulses, fall) and using appropriate values of parameters for each phase. The true angle of motion of the Anchor (12°) with respect to that of the Balance (270°= 4.7124 [rad]) can also be taken into account. Here a clear and simplified example of calculation is given, which, however gives a sufficient estimate of this energy loss.
The simple summing of the weight of the Balance to the lateral impulse forces on its staff is also extreme and somewhat unrealistic. This evaluation could be also refined on the basis of statistical data on the movements of the watch during use.
Now, also the assumption can be made that, at least for some period of time, there is no contact friction between Balance staff and its bushing and that a situation of fluid-dynamic staff support is established.

In this case, the tangential viscous force per unit area of the staff surface is:

$$\tau = \mu \frac{dV}{dr} [\frac{N}{mm^2}] \qquad (20)$$

The tangential force on the staff is then $F_t = \tau S [N]$ (21)
Where S is the portion of the staff surface on which τ acts.

The following data are here assumed:
$-\mu=\nu \rho$ is the kinematic viscosity of the oil and since ν is assumed equal to 150 [mm^2/s] and the relative density of oil is assumed 0.97, μ = 150 * 10^{-9} = 1.5 10^{-7} [N s /mm^2]

- ρ is the oil density, assumed equal to 0.97; (=0.97/98 [kg s/dm^4] = 0.97 *9.8/98 10^8 [N s/mm^4]≅ 10^{-1} *10^{-8}= 10^{-9} [N s/mm^4]
- V, the tangential speed of the staff is assumed, on the average, equal to 40[r/s], average rotational speed, times 0.1, staff radius, V= 4 [mm/s]
- dr is the gap between staff and bushing, assumed of the order of 0.005[mm]
- the surface area where τ acts is supposed to span for 90° degrees circumferentially on the staff and for 1mm in the axial direction of it S= π/2 * 0.1 * 1 = 0.16 [mm^2]

The viscous hydrodynamic couple is then :

C_h = 1.5 10^{-7} *4/0.005 *0.16 * 0.1 = 1.5*10^{-7}*4*2*10^2*1.6 10^{-1}*10^{-1}= 19.2 10^{-7} = 1.92 10^{-6} [Nmm]
And the energy lost in one period is:

$$E_{t,h} = 1.92\ 10^{-6} * 2 * 4.7124 = 1.81\ 10^{-5}\ [N\ mm] \qquad (22)$$

This lost energy is about 10% of that of a contact friction calculated above.

7.5- Energy loss due to tip of staff friction on its rest in the balance (and in the Anchor)

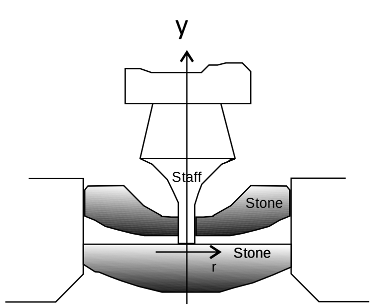

Fig. 8 Staff – Stone rest

In the case where the watch has been kept in horizontal position, the friction loss of the Balance staff happens at the circular contact surface between staff end and supporting stone (Fig. 8).

The energy lost in one period is:

$$E_{l,f} = 2 * \theta_0 * \int_0^R \frac{M_b}{\pi R^2} f * 2\pi r * rdr = 2 * \vartheta_0 * \frac{R^3}{3} * \frac{M_b}{\pi R^2} * f * 2 \pi =$$
$$= 2 * 4.7124 * 10^{-1} * 3.3\ 10^{-1} * 5.95\ 10^{-4} * 2 * 1.5\ 10^{-1} = 555.17\ 10^{-7}$$
$$= 5.6\ 10^{-5}\ [N\ mm] \quad (23)$$

Which is a smaller figure than that of Section 7.4. Between 7.4 and 7.5 alternatives, 7.4 is adopted in the conclusions.

7.6 Energy lost by the Balance for aerodynamic drag

The order of magnitude only will be calculated here since the importance of this energy loss is minor. The surfaces exposed to the aerodynamic drag are mainly the lateral surfaces of the Balance four arms and the lateral surface of the spiral in its motion towards and away from the Balance center.

The following data have been used:

-air density (mass): ρ_a =0.125 [Kg s^2/m^4]= 0.125 *9.8 /10^{12} = 1.225 10^{-12} [N s^2/mm^4]

- drag coefficient (rectangular surface perpendicular to flow): C_d= 1.17
- surface of Balance exposed to drag, S [mm^2]: 4 arms of Balance (3.4 [mm^2] + spiral 17/16 [mm^2]= 4.5 [mm^2] (the surface of the spiral has been reduced by a factor of 16=4^2 due to estimated difference in impacting air speed with respect to arms)
- average rotational speed: 74/2=37 [s^{-1}]
- effective average speed of air, V: 37 [rad/s]* 2.15 [mm]= 79.6 [mm/s]

-lever arm of the drag force (estimated): 2.8 [mm]

- angle rotated by Balance in one period: 2*270° = 9.42 [rad]

$$E_{l,a} = \frac{1}{2} C_d\, \rho_a\, S\, V^2 * 2.8 * 9.42 = 0.5\ \ 1.17\ \ 1.225\ 10^{-12}\ 4.5\ \ 79.6^2 =$$
$$= 5\ 10^{-1} * 1.17 * 1.225\ 10^{-12} * 4.5 * 6.34\ 10^3 = 204\ 10^{-10} = 2.04\ 10^2 * 10^{-10} =$$
$$2.04\ 10^{-8} [N\ mm]\ ] \quad (24)$$

Which confirms that this item is negligible.

7.7 Energy supplied to the Balance during the impulse phases in one period

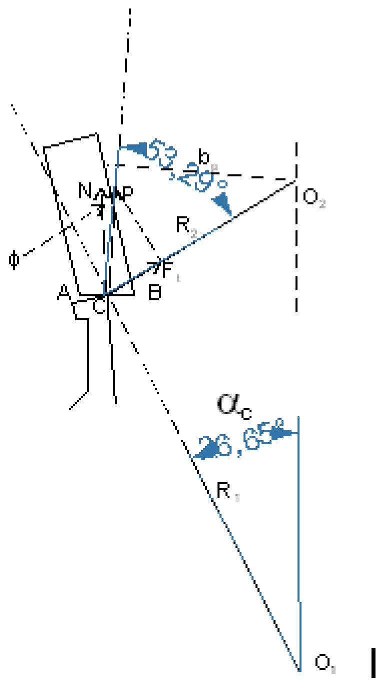

Fig. 9 Left impulse phase

From a drawing like Fig. 9. and other general data concerning power couple, it is possible to calculate the work done by the Wheel on the Anchor (E_i , which is ultimately transmitted to the Balance) during each impulse phase. Fig. 9 represents the geometrical situation in the middle of the left impulse , assumed as representing the average situation during impulse (this calculation can be made more precise by a further subdivision in phases).

The symbols and data are almost the same as in Section 7.2 plus those shown in Fig.9. The formulas are the following ones:

C_{anchor} (couple transmitted to anchor by wheel) $=P * b_p$; $P=F_t/\cos(90°-\alpha_c-\phi)$;

$F_t= C_w/R_1$ (see (18)); $E_i=P\ b_p\lambda_i$ (work during impulse of angular amplitude $\lambda_i=8.5°=0.148[rad]$) (25)

$f=\tan(\phi)$ is assumed here as in 7.2 equal to 0.2; then ϕ is 0.197 [rad], 11.29° .

It is assumed, as in equidistant escapements, that $R_2/R_1=\tan(\alpha_0)=0.577 \cong \tan(\alpha_c)$

The scale of the drawing in Fig.9 can be obtained by the knowledge of the distance between O_1 and O_2 = 3.15 [mm]; b_p is then 1.161 [mm] and R_1 is 2.7373[mm].

$P=8.38\ 10^{-4}$ (see (18)) $/\cos((90-26.65-11.29)/57.2958)= 8.38\ 10^{-4}/6.15\ 10^{-1}= 1.35*10^{-3}$ [N]

$C_{anchor}= 1.35*10^{-3}* 1.161=1.57\ 10^{-3}$ [N mm]

$E_i=C_{anchor} * \lambda_i = 1.57\ 10^{-3} *8.5/57.2958 =1.57\ 10^{-3} * 1.5\ 10^{-1}=2.355\ 10^{-4}$ [N mm]

In one period the energy gain is put equal to the double for simplicity (in reality the exit impulse couple is higher than the left one because its lever arm is larger [Reymondin]).

$E_i =4.71\ 10^{-4}$ [N mm]

7.8 Summary of energy balance and consequences

Energy loss or gain in one Balance oscillation period (0.4 s) [N mm]	Loss	Gain	Percentage of Balance energy
E. lost during unlocking	$-1.46\ 10^{-5}$		$1.76\ 10^{-5}$ $/5.48\ 10^{-3}$ = 0.27%
E. lost during initial shocks	$-3.82\ 10^{-6}$		negligible
E. lost by friction in the Balance staff	$-2.03\ 10^{-4}$		3.7%
E. lost at the tip of staff (friction)	$(-5.6\ 10^{-5})$		1%
E. lost for aerodynamic drag	$-2.04\ 10^{-8}$		negligible
E gain during impulse phases		$4.71\ 10^{-4}$	8.6 %
TOTAL	$-2.176*10^{-4}$	$4.71\ 10^{-4}$	
Δ E =Difference = $2.534\ 10^{-4}$ [Nmm]			4.6%

Δ E = 1/2 J_b ($\omega^2_{w2} - \omega^2_{w1}$); ω^2_{w2}= 2* 2.534 10^{-4}/2 10^{-6}+74.0224^2 = 253.4+ 5479 = 5732[/s^2]; ω_{w2} = 73.71 [/s]

$d\theta_0 = d\omega_{wmax}/\omega_0$ = (75.71–74.0224) /15.708 = 1.68/15.708= 0.11 [rad] =6.16°

It can be concluded, with the possibility to perform more precise calculations as indicated above, that the Balance will adjust to a maximum angle of oscillation different by a small angle from the initially assumed one.

This is a rather convincing indication of the realistic nature of the calculation method here suggested.

For clarity, the change of Balance amplitude of oscillation can compensate moderate unbalances in energy losses and gains. The energy loss is, in fact, essentially proportional to Balance amplitude, while energy gain is not (fixed position of Banking Pins). Similarly, any moderate change in the power couple C_w can be compensated by a change of Balance oscillation amplitude. Moreover, any moderate change in oscillation amplitude does not affect oscillation period. This escapement, then, is self-adjusting in front of energy unbalances (which can be originated by unbalances between Balance energy losses and gains or by insufficient watch motive power.

7.7 Quality factor

For supporters of this factor, the result here is:

$$Q = \pi \, \frac{stored\ vibration\ energy\ (Balance)}{vibration\ energy\ lost\ per\ half\ cycle} = \pi \frac{5.48\ 10^{-3}}{1.09\ 10^{-4}} = 158$$

8- Final Considerations emerging from the preceding treatment

- It is usually said that time "is the indefinite continued progress of existence and events that occur in apparently irreversible succession from the past to the present to the future". ".... it may be that there is a subjective component to time, but whether or not time itself is "felt", as a sensation, or is a judgment, is a matter of debate." There are, however, different views of time in physics, philosophy and religion. Time perception varies according to the phenomena under consideration. We speak of physical time, biological time, continuous and quantized time. It could be said that the time perceived by an observer of a lever escapement mechanism in operation is a quantized time, a more scientific expression than that of a "stop and go" process.
- The " time showing" part of an anchor lever escapement (starting with the Wheel) stays stopped for about 90% of the time; since an intentionally stopped time keeping device is absolutely precise

(indeed, only one possible speed, namely zero speed, exists), this feature adds to the overall watch precision

- As mentioned in Section 7, the Balance oscillator is nearly “isochronous” (as in other escapement mechanisms) so its oscillation period is not significantly influenced by its amplitude; moreover, the whole escapement is self – adjusting, namely any unbalance in its energy is firstly compensated by a variation of the Balance amplitude of oscillations with no variation of the oscillation period: this features are fundamental for the precision of the escapement.
- A large value of the oscillation amplitude θ_o and Wheel impulses located at a small Balance oscillation angle θ, contribute to the attenuation of the disturbances to the oscillation Period (see Airy’s formula (4)).
- Finally a rather low level remark: the lever anchor escapement is rather flat and so it is suitable for wrist watches.

9- APPENDIX

FUNCTIONING AND FEATURES OF THE SWISS LEVER ANCHOR ESCAPEMENT

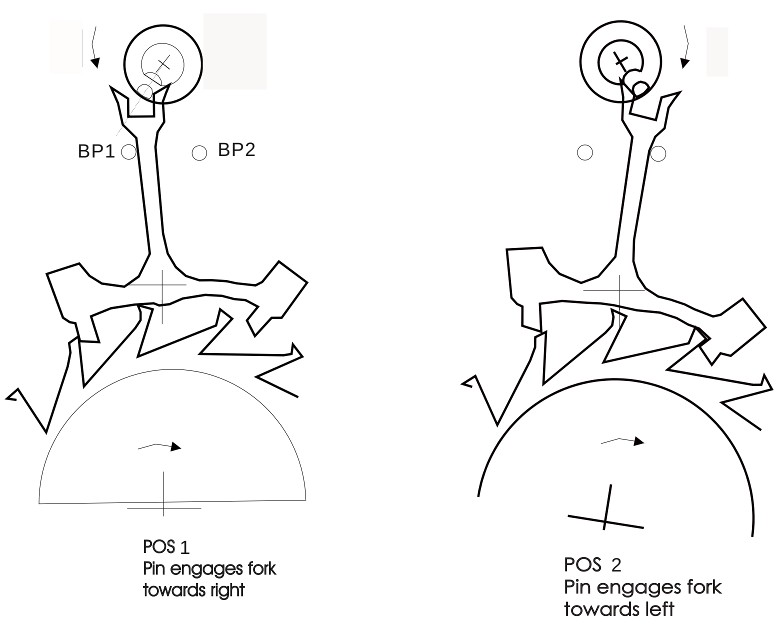

Fig. 10 The two Wheel blockage positions during the escapement motion.

In this escapement type, two mechanical parts transmit energy from the escapement Wheel (and ultimately from the power spring of the watch) to the Balance: the Anchor with its fork and the jewel pin of the Balance.

The Balance acts as the regulator of the Wheel (and of the watch hands) movement. The energy lost by the Balance (friction and shocks principally) is compensated by the impulses transmitted to it by the Wheel via the Anchor.

In the instants immediately before the position 1 of Fig. 10 , while the Balance freely moves counter clockwise running through one of its, so called, supplementary angles without contact with the Anchor, one of the Wheel teeth rests on the "rest plane" of the Anchor "entrance" pallet. The Wheel is blocked as all other components of the escapement except the jewel pin which is part of the Balance. The Wheel can be alternatively stopped either by the left pallet or by the right pallet of the Anchor. The Anchor is stopped by one of the block pins BP1 and BP2.

In these conditions the Balance rotates counter-clockwise (assumed as positive direction for the Balance movement) and at a certain moment the jewel pin enters in the Anchor fork and forces it to rotate around the center O_2 towards right. The left pallet is freed and the previously blocked left tooth of the Wheel which is now free to move, under the power continuously transmitted by the watch power spring, in the clockwise (for the Wheel and Anchor positive) direction. The Wheel is free for a short time only since it immediately meets the impulse plane of the entry (left) moving pallet. The Wheel tooth pushes Anchor in the clockwise (positive) direction. This movement is strongly accelerated, to the point that shortly the fork attains a speed higher than that of the jewel pin and start pushing it with an inversion of roles. The Anchor impulse gives to the Balance an additional acceleration, which compensates the losses of energy due to internal resistances (friction, etc.). The Balance has oscillations of the same amplitude if the damping and the supplied energy continue to exactly compensate each other. The impulse lasts until the left tooth leaves the impulse plane of the pallet. The Wheel is again free, but its movement is stopped when the right tooth falls on the rest face of the exit pallet (right). At the same instant the fork is very close to the banking pin BP2 (right) which it will reach after having run for a small angle. The Fork, the Anchor and the Wheel are then stopped.The Jewel Pin leaves the fork and continues with the Balance its movement in a clockwise direction. After the Jewel Pin has left the fork, the Balance is completely free; it describes the "supplementary arc" (a long arc, about 270° wide, shown in Fig. 3) until the end of the Balance span, when the growing couple of the spiral stops it and forces it to invert its path. From this point on, the supplementary arc is described in a clockwise direction and the following events are as in the preceding alternance:

- entrainment of the fork and of the Anchor by the Jewel Pin and, at the same time, unlocking of the right tooth of the Wheel;
- impulse on the Anchor by the right tooth and consequent acceleration of the movement of the fork which becomes a driver and increases the speed of the Balance;
- once the impulse stops, the tooth at left falls on the left pallet and the fork is stopped by BP1 while the Balance continues its movement in a clockwise direction (supplementary angle)
- the Jewel Pin again enters the Anchor fork at left and forces the Anchor to rotate clockwise
- the above described events then repeat again.

10- SYMBOLS AND DATA USED

b_p, lever arm of impulse force P on Anchor (1.61 [mm])

BP, Banking Pins

C_{anchor}, couple on Anchor during impulse (7.5)

$C_{h,}$ viscous hydrodynamic couple on Balance staff

$C_{w,}$ Couple transmitted to the Wheel by the Gear Train, $2.287*10^{-3}$ [N mm] [form [Vermot] CD, Data]

D, Balance Jewel Pin, Fig. 2

Equidistant escapement, escapement in which the locking faces of the two pallets are at the same distance from the anchor staff

E_i , energy transmitted from Wheel to Anchor and to Balance during the impulse phase

$E_{l,f}$ energy lost in the Balance and Anchor staffs for each Balance oscillation period

f, friction coefficient between Wheel tooth and Anchor pallet during unlocking (= 0.2)

F_t, tangential force transmitted from the Wheel to the Anchor pallet [N]

K, rotational elastic constant of the Balance spiral spring, 5.03 10^{-4} [N mm /rad]

J_a, moment of inertia (mass) of Anchor , 1.02 10^{-8} [N mm s^2]

J_{a+gt}, moment of inertia (mass) of Anchor + Gear Train, 2.1 10^{-8} [N mm s^2]

J_b , moment of inertia (mass) of Balance, 2 10^{-6} [N mm s^2]

J_{gt}, moment of inertia (mass) of the Wheel plus Gear Train, 2.53 10^{-8} [N mm s^2]

Jewel Pin D: pin of the balance which impacts on the anchor fork, Fig. 2

Lock: term used by horologists for the penetration, in a lever escapement, of the escape-wheel tooth **a** on the pallet-stone **b,** after the impulse on the preceding tooth.
The amount of lock is measured from the centre of the lever by the locking-angle ε (α in the drawing below) , which may vary between 1° 30' and 3°, i.e. a linear value of about 0.026 to 0.0524 mm.

M_b, Weight of the Balance, (59.5 mg)

O1, O2, O3, Centers of rotation of Wheel, Anchor and Balance (Fig. 2)

O_1O_2, segment of length 3.15 [mm]

O_2O_3, segment of length 3.4 [mm]

$\overrightarrow{O3\ O1}$ Oriented vector for the reference abscissa of equation (1)

P, force transmitted from Wheel tooth to Anchor pallet during impulse phase [N]

R, radius of the Balance staff tip (Fig. 8), 0.1 mm

R_1, radius of the contact point of Wheel and pallets with center O_1, 2.728 [mm]

T, own period of Balance oscillation, 0.4 [s]

α, angle of rotation of Wheel starting clockwise from the block left position, Pos.1 (Fig. 2)

α_0, left block angle of the Wheel (= -30°, see Fig. 2)

α_c , average angle (as seen from O_1 of the impulse to the Anchor pallet (26.65° from drawing, Fig.9)

α_p = 0.11345 [rad], first part of the left impulse to the anchor pallet (pallet impulse)

α_t = 0.06981[rad], second part of the left impulse (tooth impulse)

β_e, left pallet draw angle = 13°

δ_e, unlocking angle for the anchor as seen from O_1 (the total unlocking angle for the Wheel is the double of this value = 0.00582

ΔT_{var}, variation of the Balance period for each impact against the Anchor fork [s]

ε, unlocking angle of the Anchor as seen from O_2, 2.5°

ε_c, restitution coefficient in impacts, 0.65

θ, oscillation angle of the Balance, positive in counter-clockwise direction

θ_0 , maximum oscillation angle of the Balance (initial tentative value for “basic model” = 270°=3/2 π [rad] =4.7124 [rad])

λ_ι , angle during which the anchor receives the impulse of the Balance (= 8.5° for each alternance)

Φ, friction angle on the impulse plane, 11.29°

ρ, density (mass)

ω_{bmax}= maximum real rotational speed of the Balance 74.0224 [rad/s]

ω_0, own rotational speed of the Balance $(= 2\pi/T)$, 15.858 [rad/s]

ω_w, rotational speed of wheel [rad/s]

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